

Convex Pentagon Tilings and Heptiamonds, I

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Abstract

In 1995, Marjorie Rice discovered an interesting tiling by using convex pentagons. The authors discovered novel properties of the tilings of convex pentagon which Rice used in the discovery. As a result, many new convex pentagon tilings (tessellations) were found. The convex pentagon tilings are related to tilings by heptiamonds.

Keywords: convex pentagon, tile, tiling, tessellation, heptiamond

1 Introduction

Convex pentagons in Figure 1 can be divided into two equilateral triangles ABD and BCD , and a isosceles triangle ADE . The area of isosceles triangle ADE is equal to $1/3$ of the area of equilateral triangle ABD . That is, the convex pentagons in Figure 1 include $7/3$ equilateral triangles. As will be described later, the convex pentagon in Figure 1 can be considered as a unique convex pentagon obtained from a trisected heptiamond¹. Hereafter, the convex pentagon of Figure 1 is referred to as a *TH-pentagon*. The TH-pentagon belongs to both Type 1 and Type 5 of the known types for convex pentagonal tiles (see Figure 55 in Appendix) [1, 8, 13]. Therefore, as shown in Figures 2 and 3, the TH-convex pentagon can generate each representative tiling of Type 1 or Type 5, or variations of Type 1 tilings (i.e., tilings whose vertices are formed only by the relations of $A+D+E = 360^\circ$ and $B+C = 180^\circ$).

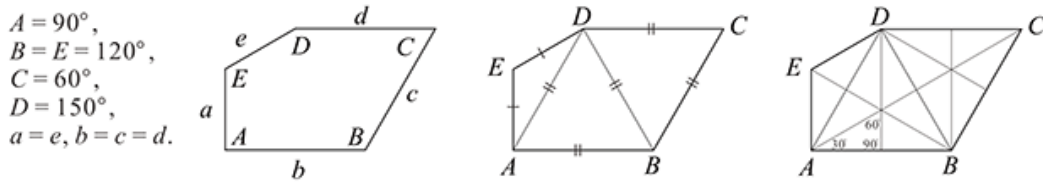


Figure. 1: TH-pentagon that belongs to both Type 1 and Type 5.

¹ A polyiamond (simply iamond) is a plane figure constructed by congruent equilateral triangle joined edge-to-edge [1, 15, 17]. A polyiamond with seven equilateral triangles is called a heptiamond or 7-iamonds. There are 24 unique pieces [1]. On the other hand, the convex pentagon in Figure 1 is also 14-polydrafter [14, 15]. However, the convex pentagon corresponding to a 14-polydrafter also exists other than the pentagon in Figure 1. It belongs to Types 1, 5, and 6 (see Figure 8(p) in [9]).

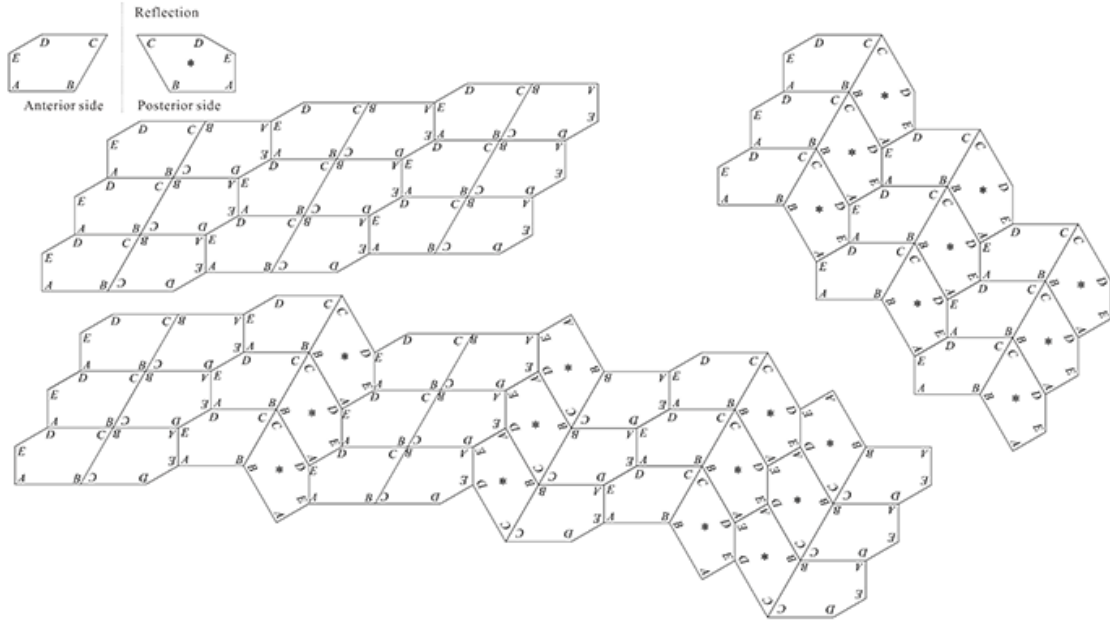


Figure. 2: Representative tiling of Type 1 and variations of Type 1 tilings.

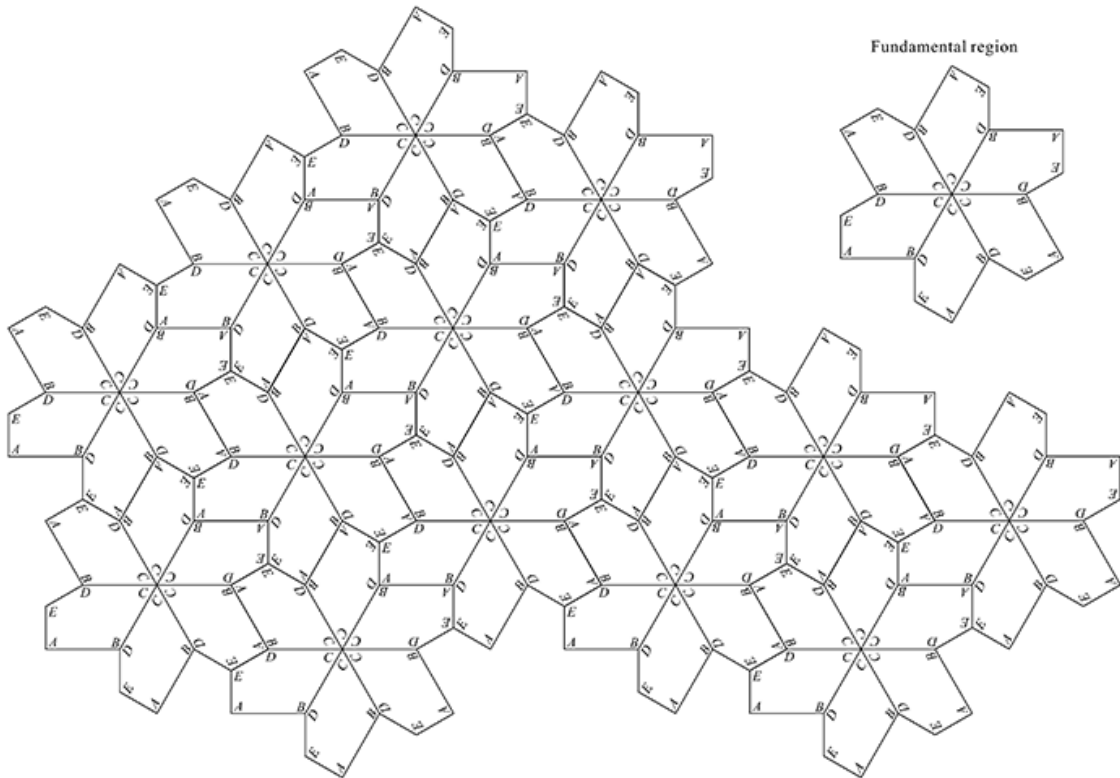


Figure. 3: Representative tiling of Type 5. The fundamental region is a unit that can generate a periodic tiling by translation only.

In 1995, Marjorie Rice discovered the interesting tiling in Figure 4 by using the TH-pentagon [5–7,13,18]. The tiling in Figure 4 decorates the floor of the Marcia P Sward lobby in the Mathematical Association of America headquarters. Hereafter, the tiling discovered by Marjorie Rice in 1995 is called a *Rice1995-tiling*. The fundamental region in Figure 4 is formed by 18 convex pentagons.

As shown in Figure 5, the authors consider a convex nonagon that is formed by three TH-pentagons. The convex nonagon in Figure 5 is called a *convex nonagon unit* (CN-unit). There are two convex nonagon units in the fundamental region of Figure 4. The outline of a convex nonagon unit is symmetrical (i.e., it is identical to the original when reflected about an axis). Therefore, when the convex nonagon units in the fundamental region of a Rice1995-tiling are reflected, the outline of fundamental region of a Rice1995-tiling is the same (see Figure 6). As shown in Figure 7, the units in a convex nonagon in Rice1995-tiling can be reversed freely [6]. Thus, from the property for convex nonagon units, the Rice1995-tiling has the property of generating nonperiodic tilings.

Although the TH-pentagon is known to be able to generate tilings of Type 1, Type 5, or Rice1995, the authors discovered that it is possible to generate tilings other than these. The newly discovered tilings (tessellations) are presented in this manuscript. Note that many proofs are omitted.

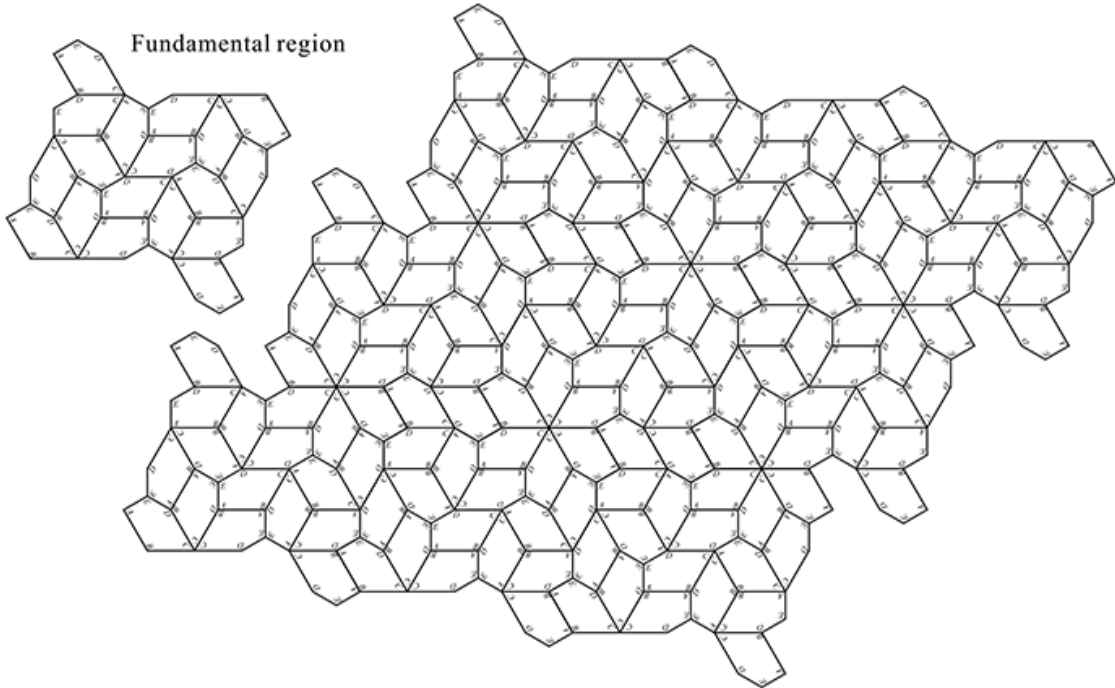


Figure. 4: Rice1995-tiling.

2 How to concentrate vertices

If the TH-pentagon generates tilings, it must have the concentration relation that the sum of some vertices (angles) is equal either to 180° or to 360° . First, the relations of 180° are

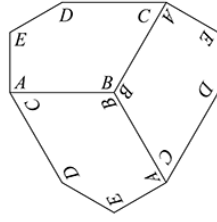


Figure. 5: Convex nonagon unit (CN-unit).

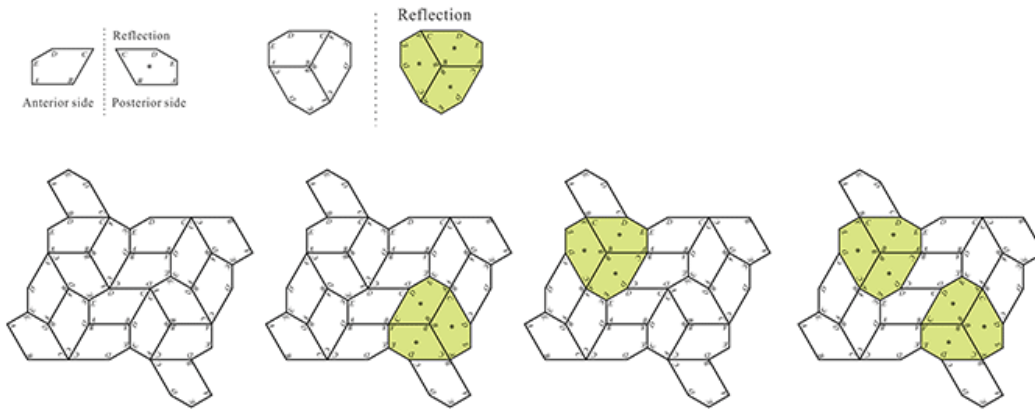


Figure. 6: Fundamental regions of a Rice1995-tiling and convex nonagon units.

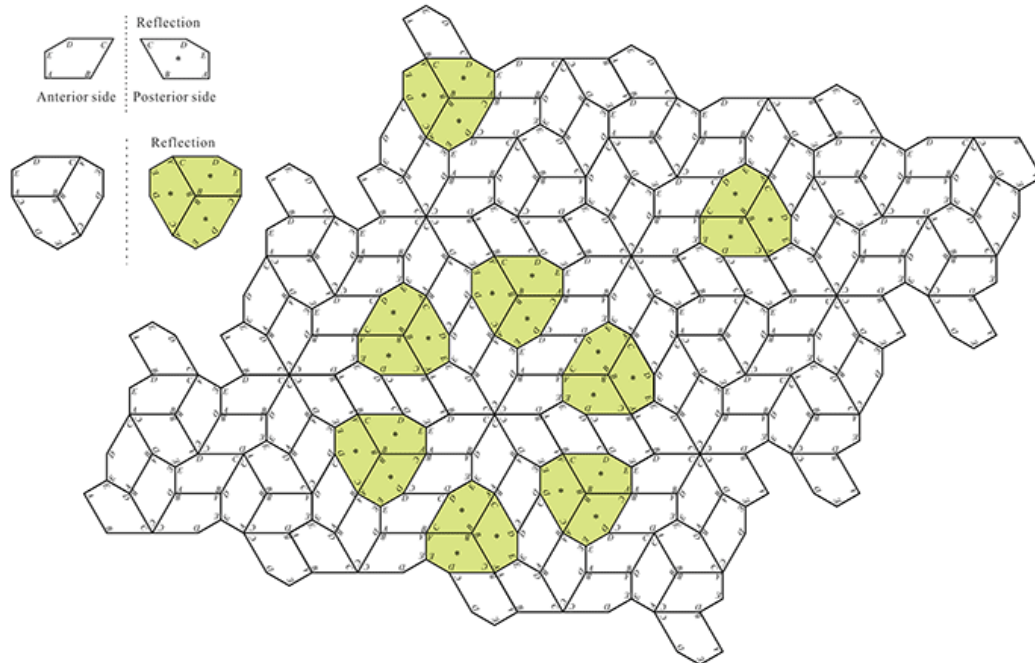


Figure. 7: Example of a Rice1995-tiling which contains reflection of convex nonagon units.

combinations of the following.

$$2A = 180^\circ, \quad B + C = 180^\circ, \quad B + E = 180^\circ, \quad C + E = 180^\circ.$$

Next, the concentration relations of 360° are combinations of the following.

$$\begin{aligned} A + B + D = 360^\circ, \quad A + D + E = 360^\circ, \quad 3B = 360^\circ, \quad 2B + E = 360^\circ, \quad B + 2E = 360^\circ, \\ C + 2D = 360^\circ, \quad B + 2E = 360^\circ, \quad 3E = 360^\circ, \quad 4A = 360^\circ, \quad 2A + B + C = 360^\circ, \\ 2A + C + E = 360^\circ, \quad A + 2C + D = 360^\circ, \quad 2B + 2C = 360^\circ, \quad B + 2C + E = 360^\circ, \\ 2C + 2E = 360^\circ, \quad 2A + 3C = 360^\circ, \quad B + 4C = 360^\circ, \quad 4C + E = 360^\circ, \quad 6C = 360^\circ. \end{aligned}$$

However, if $B + E = 180^\circ$, impossible areas are created where a TH-pentagon cannot be aligned without gaps always exist (see Figure 8). The concentration relations that cannot be used in tilings clearly are excluded. As a result, candidates of concentration relations that will be used in tilings are as follows.

$$\begin{aligned} 2A = 180^\circ, \quad B + C = 180^\circ, \\ A + B + D = 360^\circ, \quad A + D + E = 360^\circ, \quad 3B = 360^\circ, \quad C + 2D = 360^\circ, \quad 3E = 360^\circ, \quad 4A = 360^\circ, \\ 2A + B + C = 360^\circ, \quad 2A + C + E = 360^\circ, \quad A + 2C + D = 360^\circ, \quad 2B + 2C = 360^\circ, \\ 2A + 3C = 360^\circ, \quad B + 4C = 360^\circ, \quad 6C = 360^\circ. \end{aligned}$$

Next, consider the arrangements of TH-pentagons in these candidates of concentration relations. For example, for concentration relations with three vertices like $A + D + E = 360^\circ$, there are eight possible arrangements to assemble the three pentagons since each of three pentagons can be reversed (see Figure 9). Except for arrangements that clearly cannot be used in tilings, the arrangements of concentration relations with three vertices are nine patterns in Figure 10, not distinguishing reflections and rotations (i.e., the number of unique patterns in this case is nine). The other cases are also obtained similarly (see Figure 11). For example, for $2A + C + E = 360^\circ$, two patterns AACE-1 and AACE-2 as shown in Figure 11 are obtained.

Here, consider the concentration $A + D + E = 360^\circ$, $3E = 360^\circ$, and $2A + C + E = 360^\circ$ including the vertex E . In the case of $2A + C + E = 360^\circ$, there are two patterns AACE-1 and AACE-2 in Figure 11, but it is confirmed that an impossible area appears by accumulating the step by step possible arrangements of TH-pentagons. Therefore, if the TH-pentagon generates a tiling, the vertex E belongs to $A + D + E = 360^\circ$ or $3E = 360^\circ$. Furthermore, the authors confirmed that $A + D + E = 360^\circ$ and $3E = 360^\circ$ cannot coexist within the tiling.

Therefore, the TH-pentagon can generate tilings with $A + D + E = 360^\circ$ and tilings with $3E = 360^\circ$. The tilings with $A + D + E = 360^\circ$ are always variations of Type 1 tilings as shown in Figure 2. The tilings with $3E = 360^\circ$ contain the patterns EEE-1 or EEE-2 as shown Figure 10. Hereafter, EEE-1 and EEE-2 are referred to as a *windmill unit* and a *ship unit*, respectively. As shown in Figure 12, the windmill unit and the ship unit can be considered as heptiamonds. That is, the TH-pentagon is a convex pentagon that can be obtained by dividing the two types of heptiamond of the windmill unit and the ship unit into three equal parts. The convex pentagon that can be obtained by trisecting 24 types of heptiamond does not exist other than the TH-pentagon. Therefore, tilings with $3E = 360^\circ$ by TH-pentagons are equivalent to tilings of these two types of heptiamonds.

The authors discovered new convex pentagon tilings using windmill units and ship units; they are presented in the following sections.

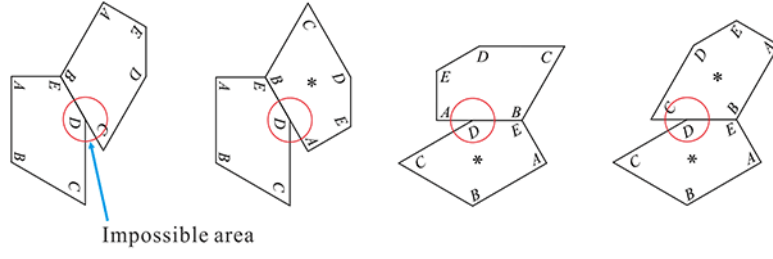


Figure. 8: Example of impossible areas where TH-pentagon cannot be aligned without gaps.

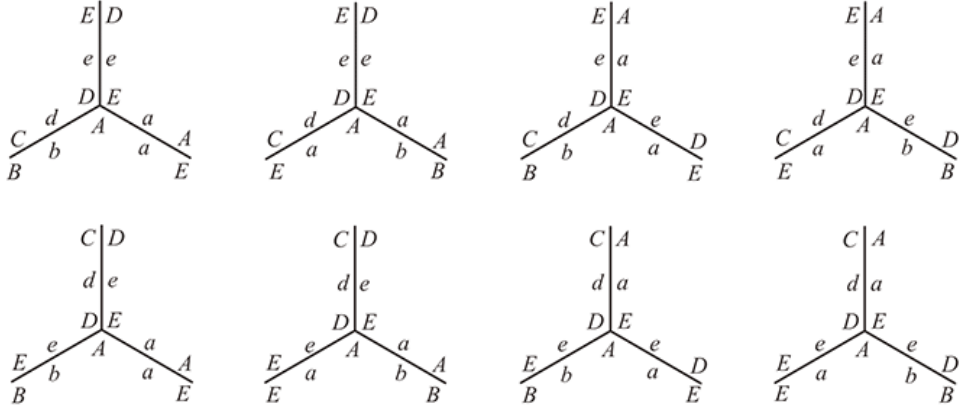


Figure. 9: Eight possible arrangements of $A + D + E = 360^\circ$.

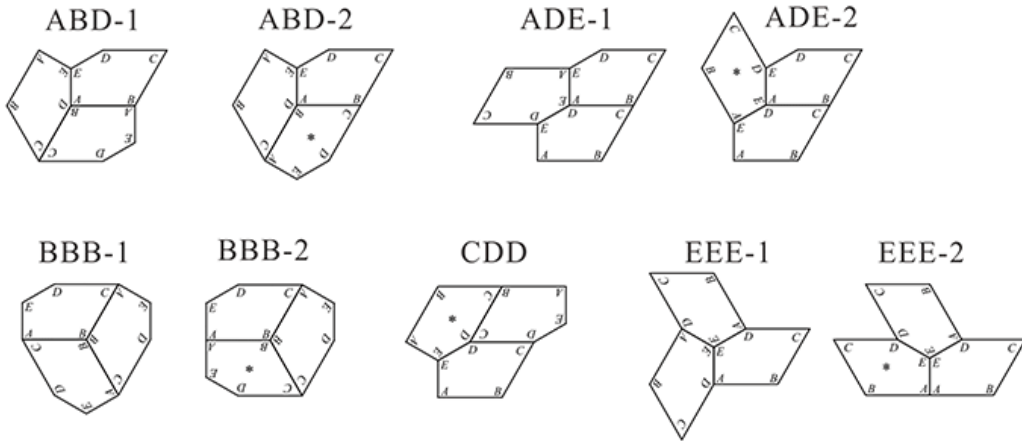


Figure. 10: Unique patterns of concentration relations with three vertices.

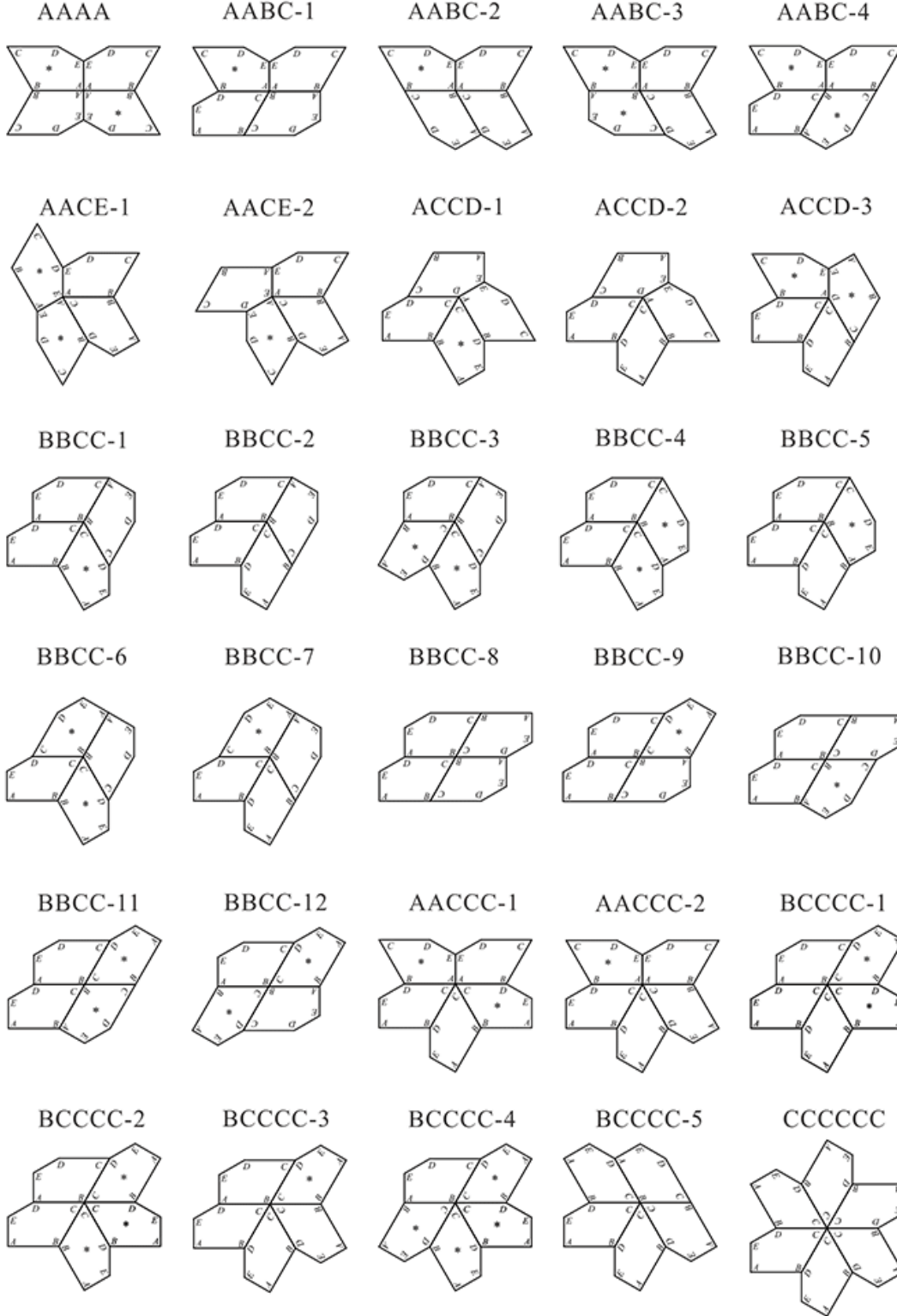


Figure. 11: Unique patterns of concentration relations with four or more vertices.

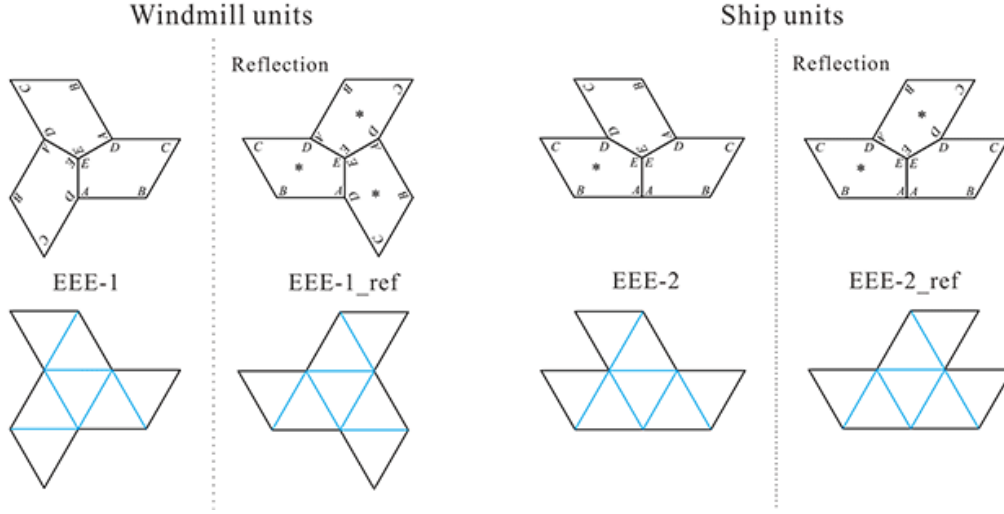


Figure 12: Windmill units and ship units.

3 Tilings by only the windmill units

In this section, tilings by only the windmill units are introduced. As shown in Figure 13, the authors consider a shape (18 sided polygon) formed by six windmill units. Hereafter, the shape is referred to as a *hexagonal flowers L1 unit* (HFL1-unit). The hexagonal flowers L1 unit has an edge in common with edge b , c , or d of a TH-pentagon.

The representative Type 5 tiling (Figure 3) and the Rice1995-tiling (Figure 4) can be generated using the hexagonal flowers L1 units. As shown in Figures 14 and 15, the representative Type 5 tiling and the Rice1995-tiling that does not contain reflected convex nonagon units are generated by the difference in the contiguity method of hexagonal flowers L1 units.

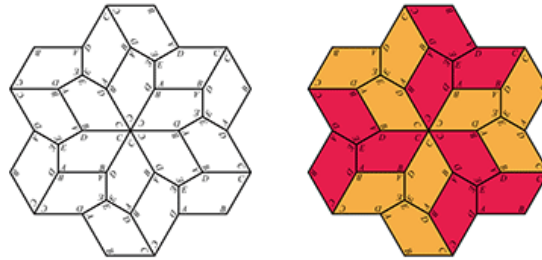


Figure 13: Hexagonal flowers L1 unit.

The hexagonal flowers L1 unit and the reflection of the hexagonal flowers L1 unit have identical outlines (see Figure 16). The eighteen convex pentagons in the hexagonal flowers L1 units in Figure 13 are anterior since the pentagon in Figure 1 is considered to be forward facing. Hereafter, the hexagonal flowers L1 unit formed by 18 anterior TH-pentagons is referred to as an *AHFL1-unit*, and the hexagonal flowers L1 unit that is formed by 18 posterior TH-pentagons is referred to as a *PHFL1-unit*. In addition, the convex nonagon unit (see Figure 5) formed by three anterior TH-pentagons is referred to as an *ACN-unit*, and the convex nonagon unit formed by three posterior TH-pentagons is referred to as a *PCN-unit*.

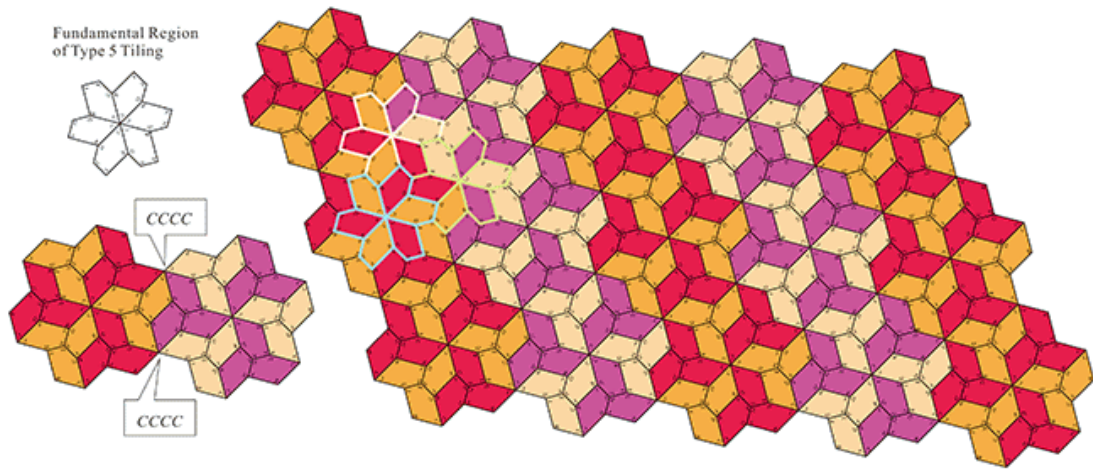


Figure. 14: Relation of representative Type 5 tiling and hexagonal flower L1 units. Note that, in order to make a contiguity method intelligible, the hexagonal flowers L1 units of different colors are used.

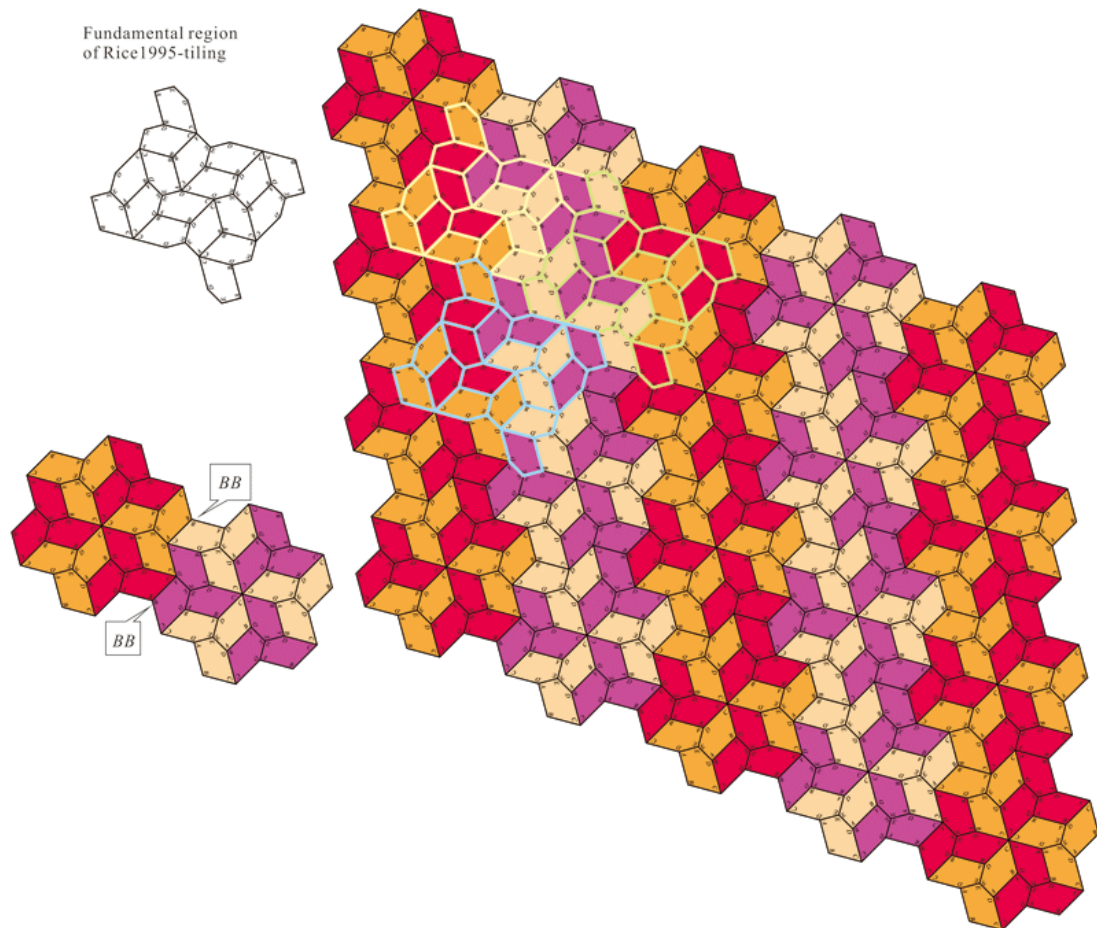


Figure. 15: Relation of Rice1995-tiling and hexagonal flowers L1 units. Note that, in order to make a contiguity method intelligible, the hexagonal flowers L1 units of different colors are used.

Figure 17 is a tiling consisting of a pair of AHFL1-unit and PHFL1-unit. (The tilings in Figure 17 have 5-valent vertices “ $B + 4C = 360^\circ$.”) The contiguity method of AHFL1-units and PHFL1-units is only a pattern as shown in Figure 18.

Since the AHFL1-unit and the PHFL1-unit have identical outlines, a hexagonal flowers L1 unit in the Type 5 tiling and the Rice1995-tiling can be changed to a reversed hexagonal flowers L1 unit freely. Figure 19 is a tiling that incorporates PHFL1-units as the basis of a Type 5 tiling that is formed by AHFL1-units. Figure 20 is a tiling that incorporates PHFL1-units as the basis of a Rice1995-tiling that is formed by AHFL1-units.

As shown in Figure 21, points concentrating six vertices C in the Type 5 tiling are joined with lines of three kinds (solid line, dashed line, and one-dot chain line). The points concentrating six vertices C exist on the intersections of the same lines. From Figure 21, it can be seen that there exist three types of points concentrating six vertices C .

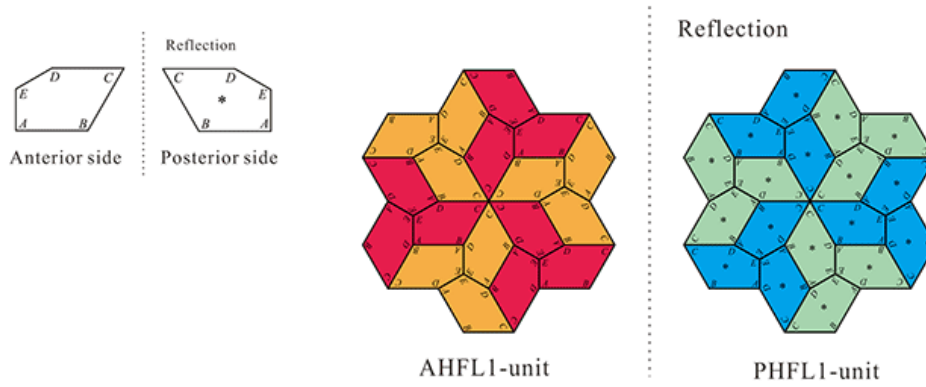


Figure. 16: AHFL1-unit (anterior hexagonal flowers L1 unit) and PHFL1-unit (posterior hexagonal flowers L1 unit).

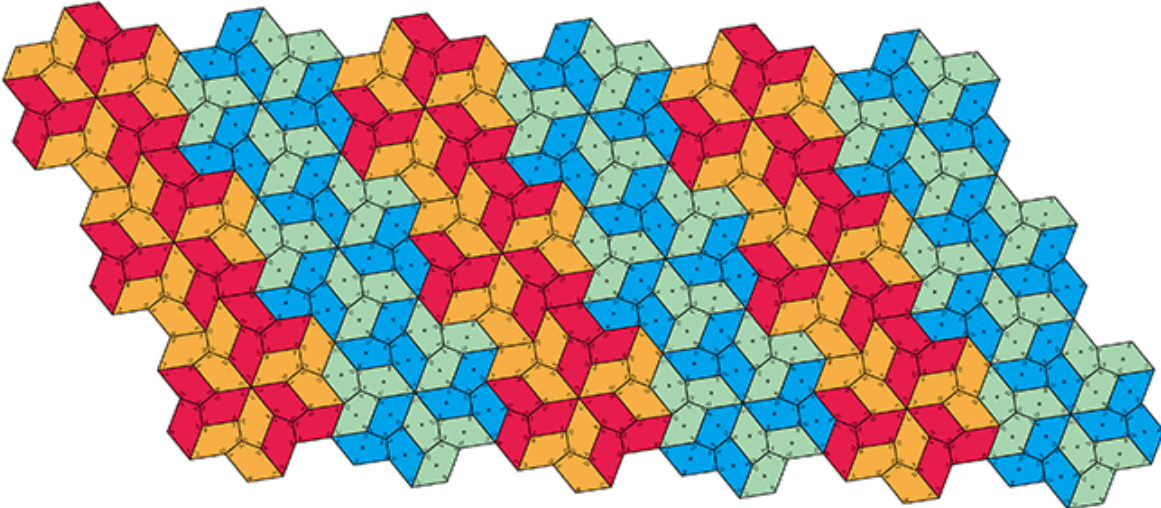


Figure. 17: Tiling by a pair of an AHFL1-unit and a PHFL1-unit.

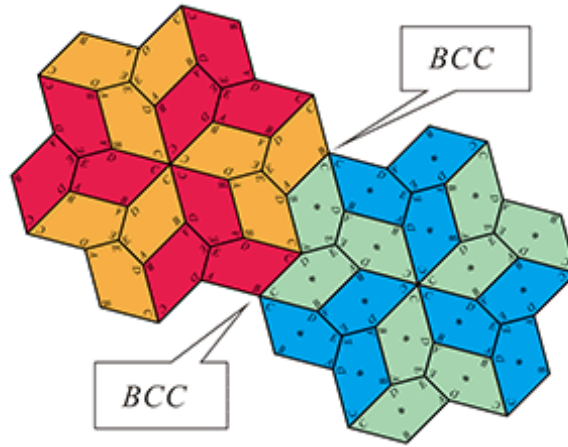


Figure. 18: Contiguity method of an AHFL1-unit and a PHFL1-unit.

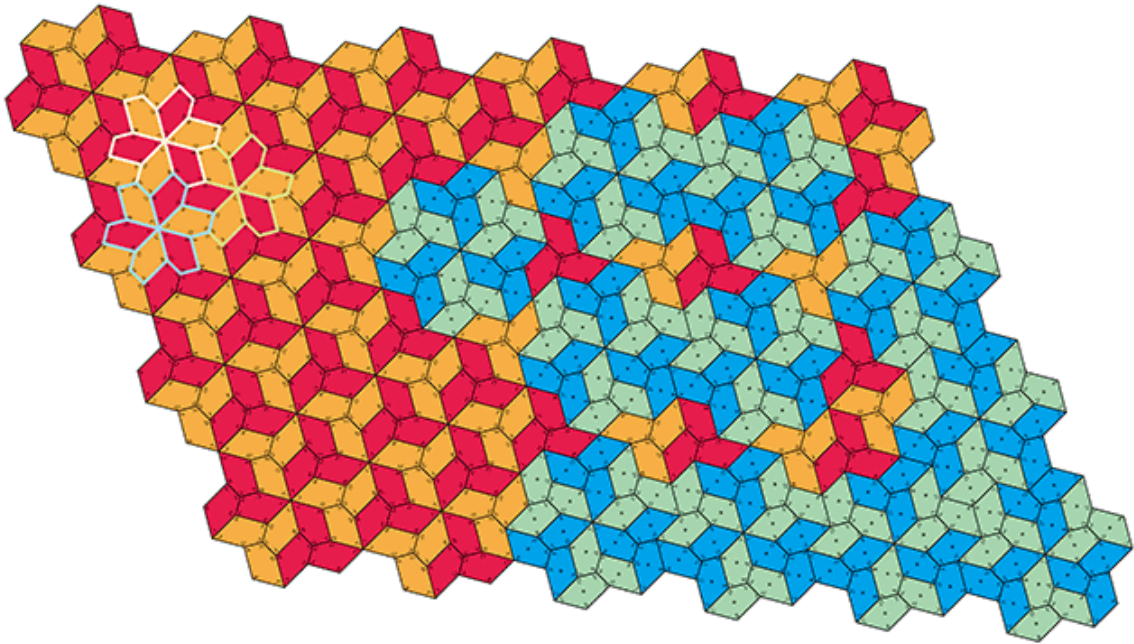


Figure. 19: Tiling that incorporates PHFL1-units as the basis a Type 5 tiling that is formed by AHFL1-units.

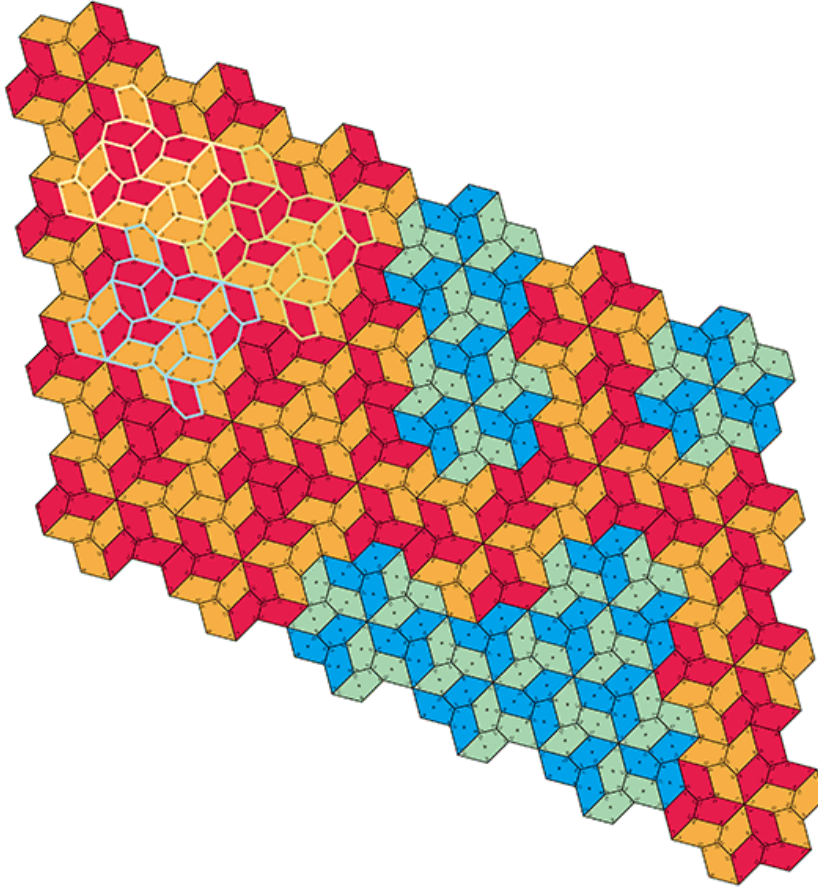


Figure. 20: Tiling that incorporates PHFL1-units as the basis of an Rice1995-tiling that is formed by AHFL1-units.

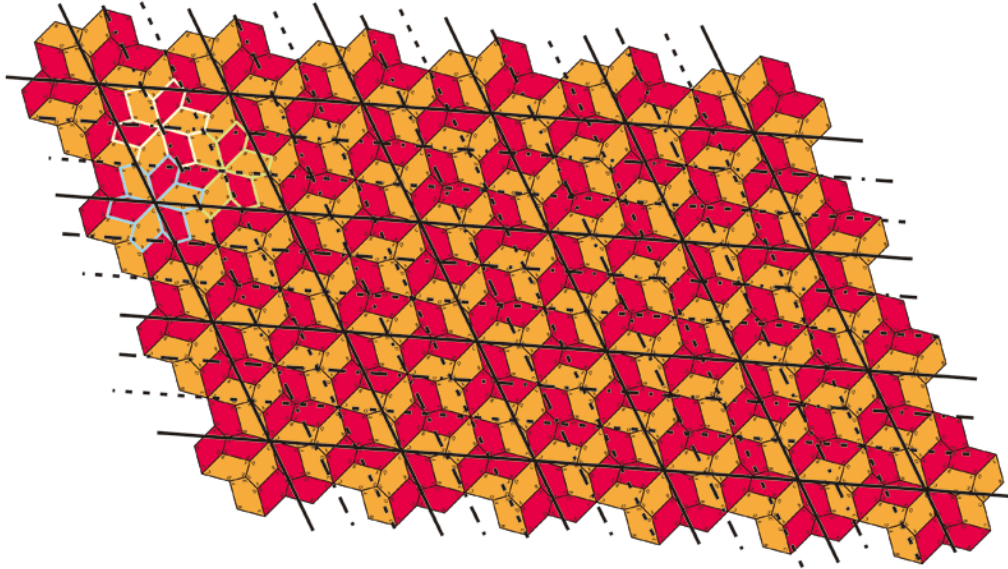


Figure. 21: Position relation of points concentrating six vertices C in the Type 5 tiling.

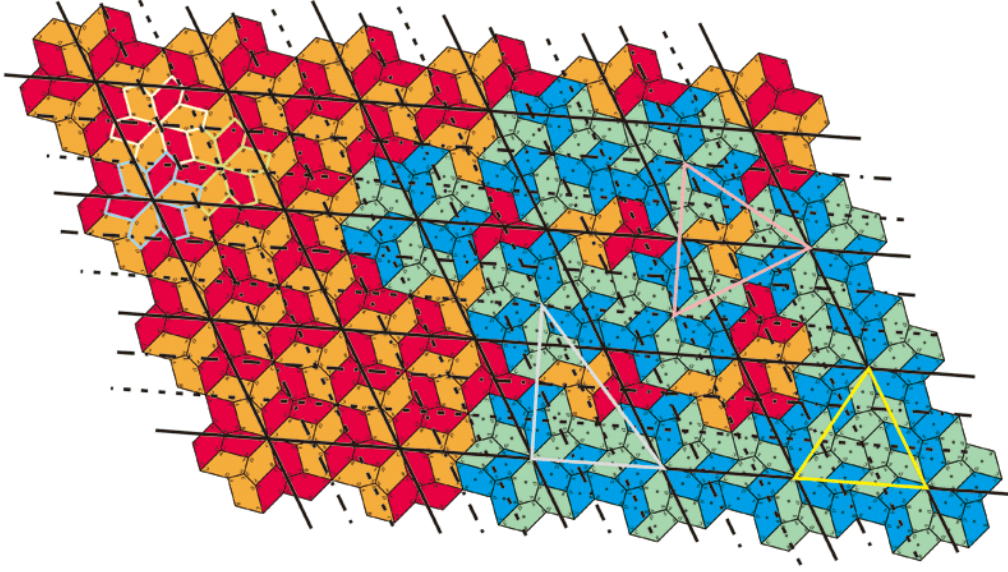


Figure. 22: Position relation of points concentrating six vertices C in the tiling of Figure 19.

Figure 22 transposes the tiling of Figure 19 to the tiling of Figure 21. From Figure 22 it is apparent that three possible relations exist between centers of three adjoining PHFL1-units, as follows:

Case 1: The centers of three adjoining PHFL1-units are on the intersections of the lines of one kind (see the yellow triangle in Figure 22).

Case 2: The centers of three adjoining PHFL1-units are on the intersections of the lines of two kinds (see the gray triangle in Figure 22).

Case 3: The centers of three adjoining PHFL1-units are on the intersections of the lines of three kinds (see the pink triangle in Figure 22).

If a Type 5 tiling with AHFL1-units is filled by PHFL1-units according to the properties of Case 1, the Type 5 tiling becomes an Rice1995-tiling with PHFL1-units (see Figure 23).

As for Cases 2 and 3 in Figure 22, it is evident that the unit is formed by a hexagonal flowers L1 unit and two reverse windmill units, as shown in Figure 24. By using the units in Figure 24, the tilings in Figure 25 (the properties of Case 3) or Figure 26 (the properties of Case 2) can be formed. The tilings as shown in Figures 27 and 28 can also be formed by applying the property of Case 2. That is, one can arrange the width of the reversed windmill units' band or arrange it in a V shape.

In addition, tilings having the properties of Cases 1, 2, and 3 can also be formed. Figure 29 is an example of a tiling with the properties of Cases 1, 2, and 3.

Thus, the TH-pentagon admits an infinite variety of periodic tilings and nonperiodic tilings by the windmill units.

Concentration relations of vertices A and D of the TH-pentagon in the windmill unit are addressed as follows. When using only the windmill unit, the concentration relations of vertices A and D are $A + B + D = 360^\circ$ (ABD-1, ABD-2) or $A + 2C + D = 360^\circ$ (ACCD-1, ACCD-2, ACCD-3). That is, patterns of contiguity method between windmill units are Classes W1, W2, W3, W4, and W5 in Figure 30. Class W1 appears inside the HFL1-unit and

in a Type 5 tiling. Class W2 appears in a connection of AHFL1-unit and PHFL1-unit. Class W3 cannot be used in a tiling. Class W4 appears in a Rice1995-tiling. Class W5 appears in the combination of HFL1-unit and the two reverse HFL1-units (see Figure 24).

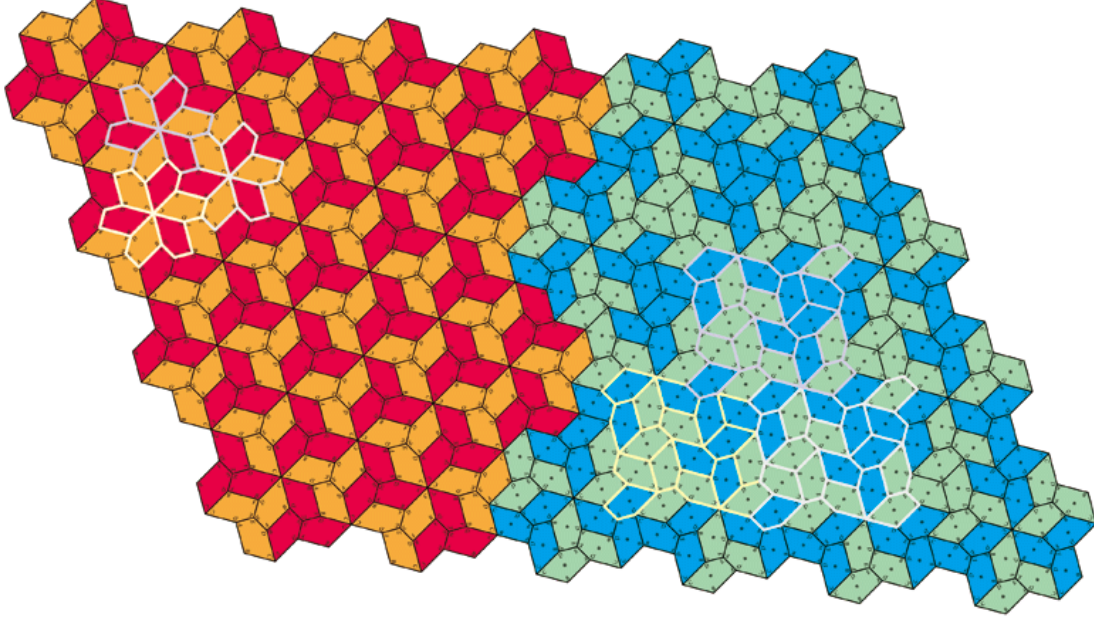


Figure. 23: Example of tiling according to the properties of Case 1.

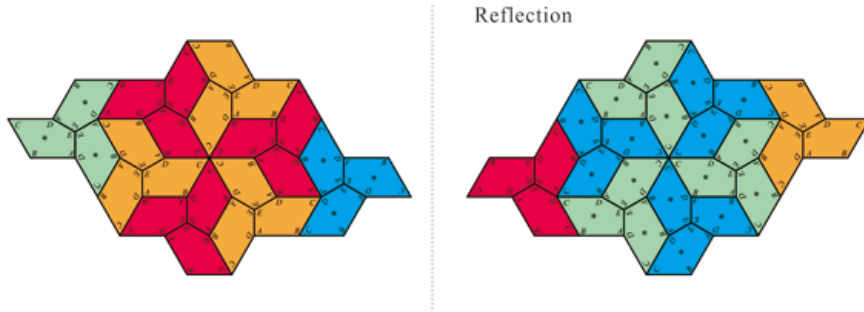


Figure. 24: Units that are formed by a hexagonal flowers L1 unit and two reverse windmill units.

4 Tilings by only the ship units

In this section, tilings by only the ship units are introduced. A concentration relation of two vertices D of the TH-pentagon in the ship unit is considered next. When using only the ship unit, the concentration relation of two vertices D is $C + 2D = 360^\circ$, and the pattern is CDD in Figure 10. Therefore, the patterns of contiguity method between ship units are Classes S1, S2, S3, S4, and S5 in Figure 31.

The tiling in Figure 32 is a tiling with Class S1. The tiling of Figure 32 has the property of freely arranging three kinds of objects like a band in one-dimension. The tiling in Figure 33

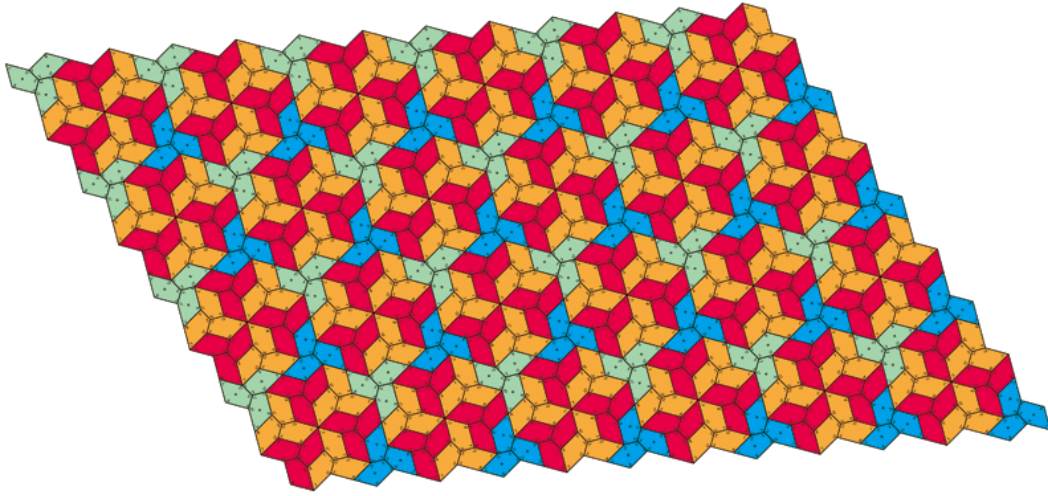


Figure. 25: Example of a tiling according to the properties of Case 3.

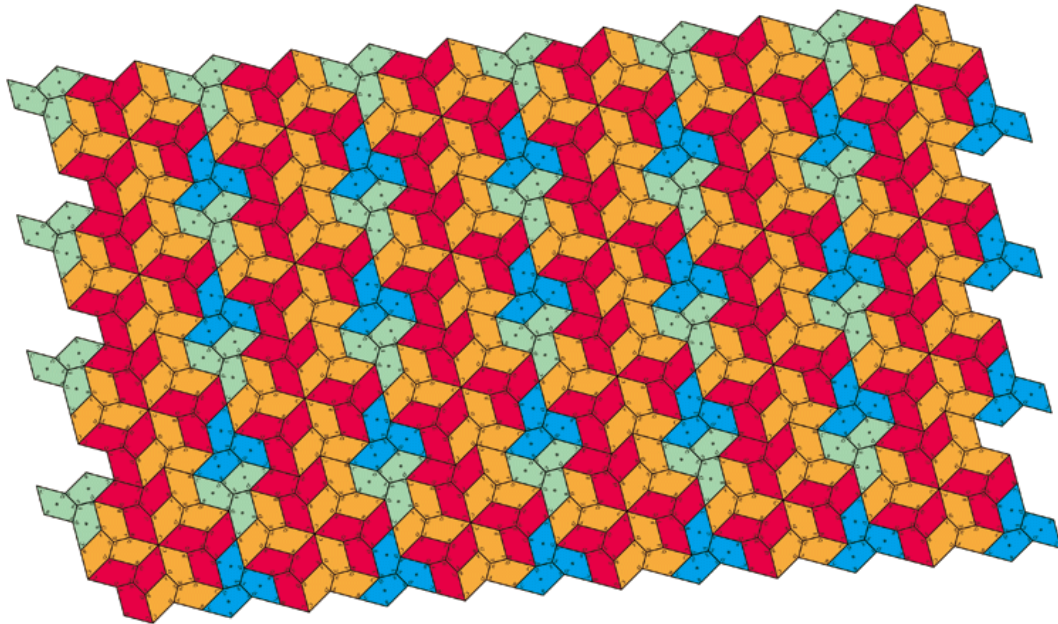


Figure. 26: Example of a tiling according to the properties of Case 2.

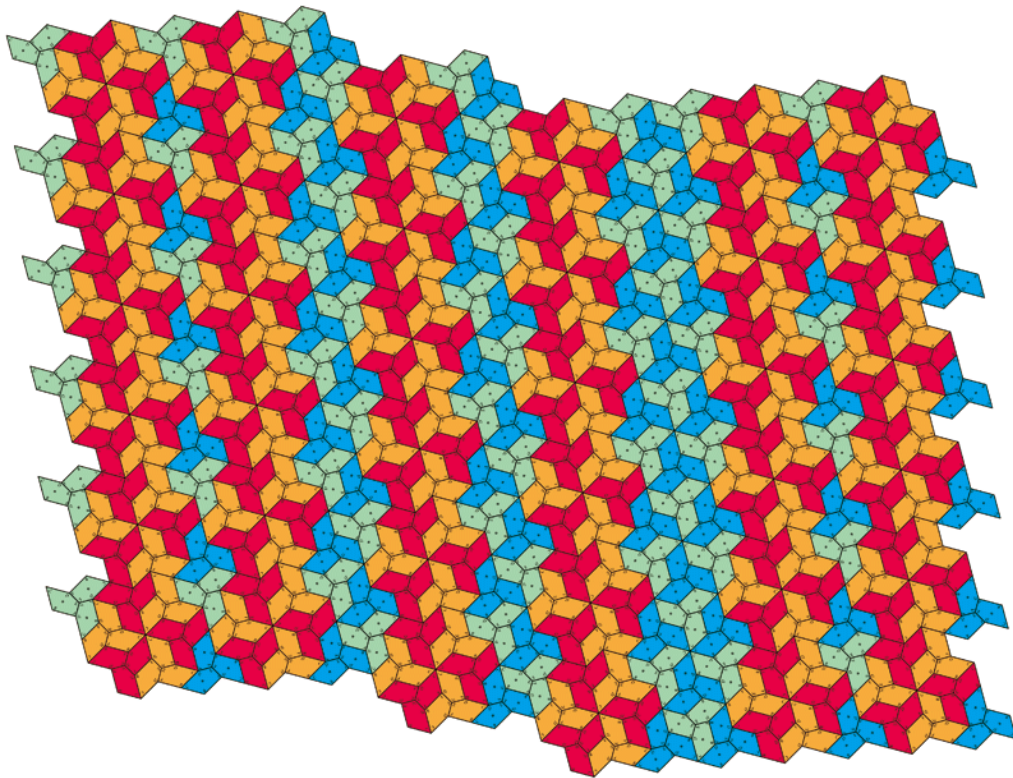


Figure. 27: Example of a tiling that is formed by applying the properties of Case 2.

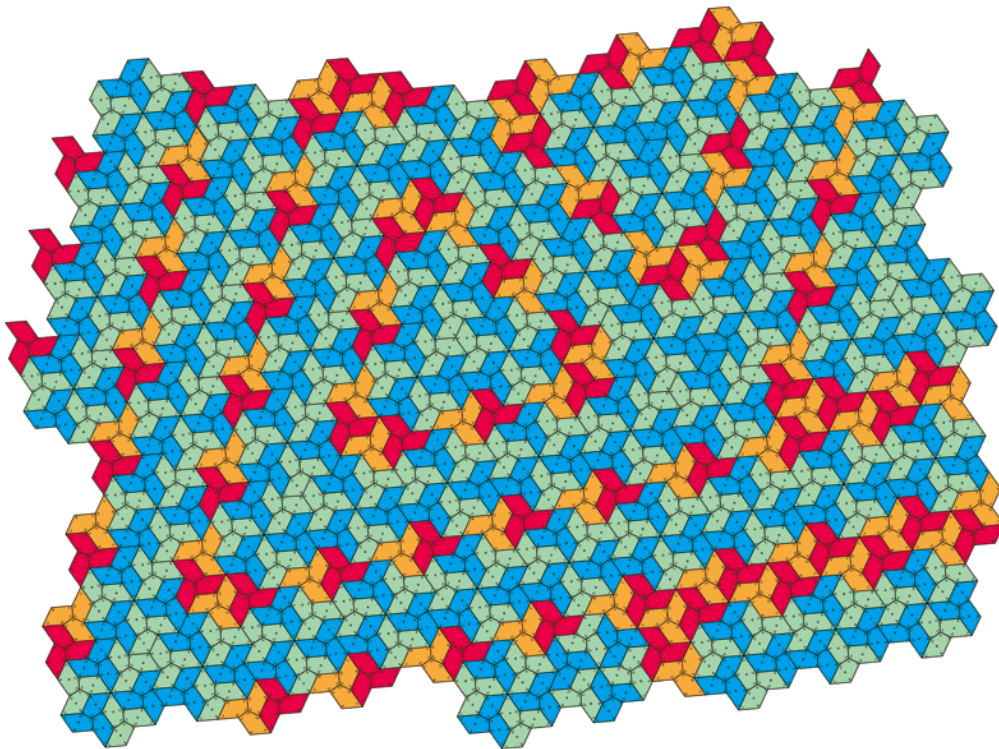


Figure. 28: Example of a tiling that is formed by applying the properties of Case 2.

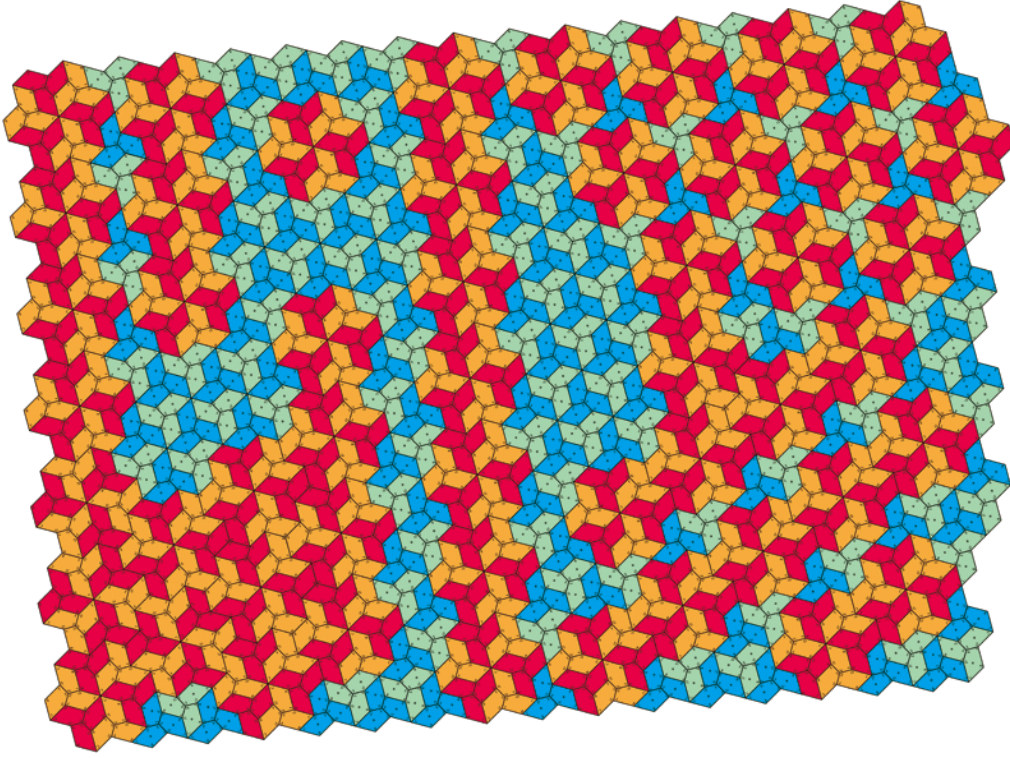


Figure. 29: Example of a tiling with the properties of Cases 1, 2 and 3.

is a tiling with Class S2, and the tiling in Figure 34 is a tiling with Class S3. The tilings in Figures 35 and 36 are tilings with Class S4. Then, the tiling of Figure 35 has the property of freely arranging base parts in one-dimension. There is no tiling with only Class S5. In addition, the tiling in Figure 37 is a tiling with Classes S1 and S3. The tiling in Figure 38 is a tiling with Classes S2 and S4. The tilings of Figures 37 and 38 have the property that base parts can combine freely in one dimension.

Those tilings are the tilings of only the ship unit the authors have currently found. (In fact, although there are other tilings, they are shown in the next section by their properties.)

Thus, the TH-pentagon admits an infinite variety of periodic tilings and nonperiodic tilings by the ship units.

5 Tilings by the windmill units and the ship units

In this section, the authors introduce the result that the TH-pentagon admits infinite variety of periodic tilings and nonperiodic tilings by the windmill units and the ship units.

5.1 Rice1995-tiling with ACN-units and PCN-units

The Rice1995-tiling in Figure 4 is a Rice1995-tiling by the anterior TH-pentagon, which is considered as tiling by only the anterior windmill unit. The Rice1995-tiling in Figure 39 contains a PCN-unit. Apparently, it is equal to replacing a windmill unit with a ship unit (or vice versa) when a CN-unit is reversed. Therefore, the Rice1995-tilings with ACN-units

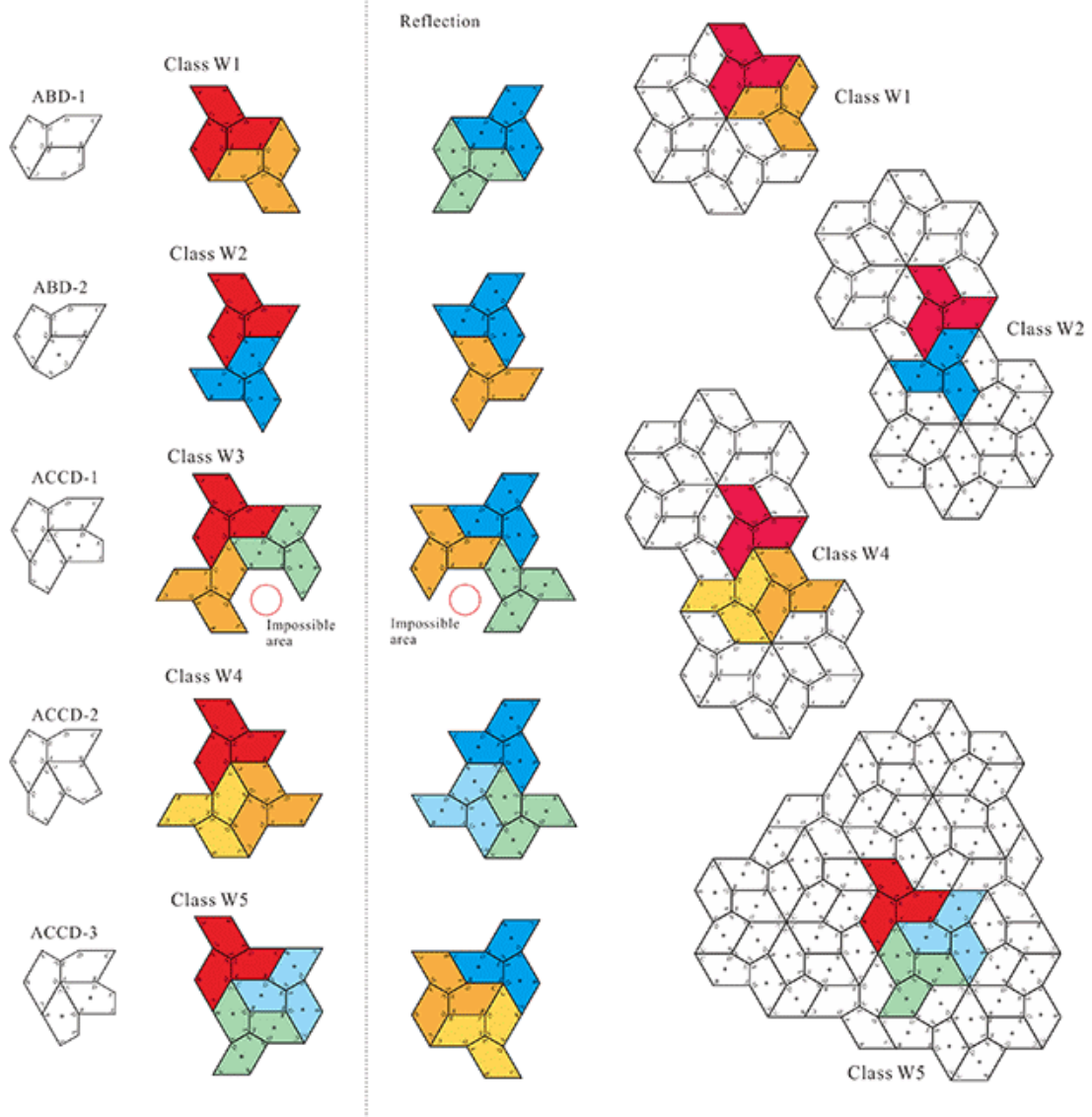


Figure. 30: Contiguity method of windmill units.

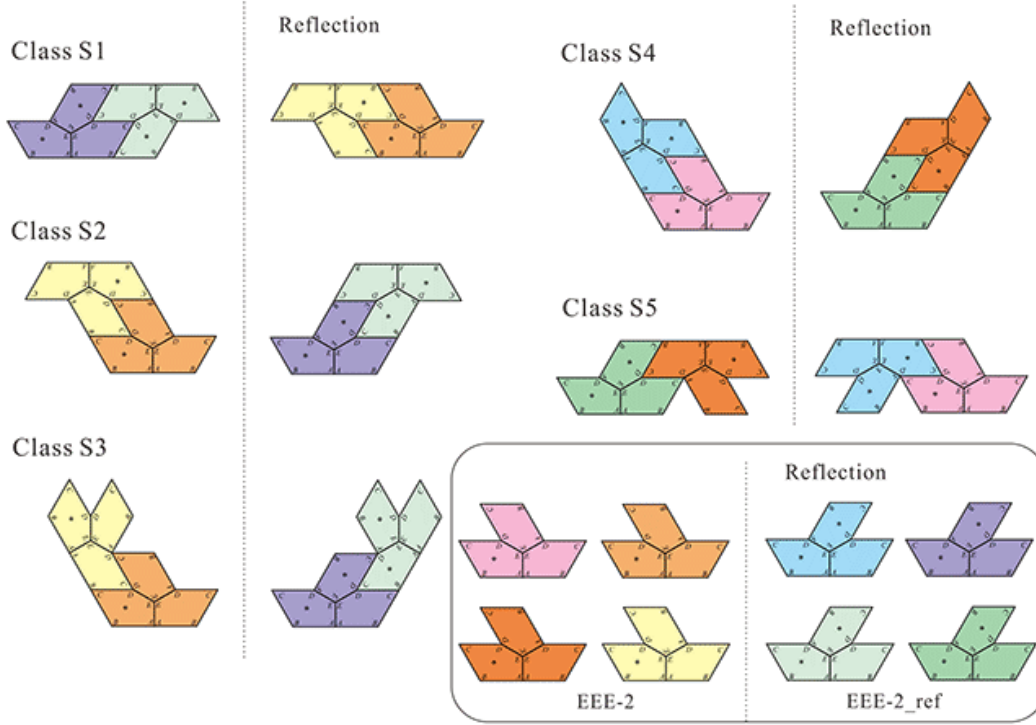


Figure. 31: Contiguity method of ship units.

and PCN-units as shown in Figures 7 and 39 are tilings by the windmill units and the ship units.

5.2 Consideration of similar hexagonal flower units

In Section 3, the hexagonal flower unit (i.e., SHL1-unit) formed by six windmill units was presented. As shown in the next subsections, by using the windmill units and ship units, it is possible to create a similar hexagonal flower unit with the side length of two or three. On the other hand, the authors have confirmed that there are no hexagonal flower units with the side length of four or more. The reason is briefly described below. Focusing on the convex portion of the hexagonal flower unit, one obtains combinations of windmill units and ship units capable of forming the convex portion. In addition, the vertex of 120° in the convex portion will use $B = 120^\circ$ or $C + C = 120^\circ$. As a result, the combinations of which the side length is three or more and will generate the tiling are three patterns (not distinguishing reflections and rotations) that use the two ship units in Figure 40. By using these three patterns, the authors confirmed that there are no hexagonal flower units with the side length of five or more. In addition, when the side length is four, it is possible to form a part, but when trying to form hexagonal flower units with side length of four, an impossible area appears². Therefore, it is impossible to form hexagonal flower units with side lengths of four or more by using TH-pentagons.

² By also a program that used Algorithm X, the authors confirmed that there is no hexagonal flower unit with side length four [2–4, 10–12].

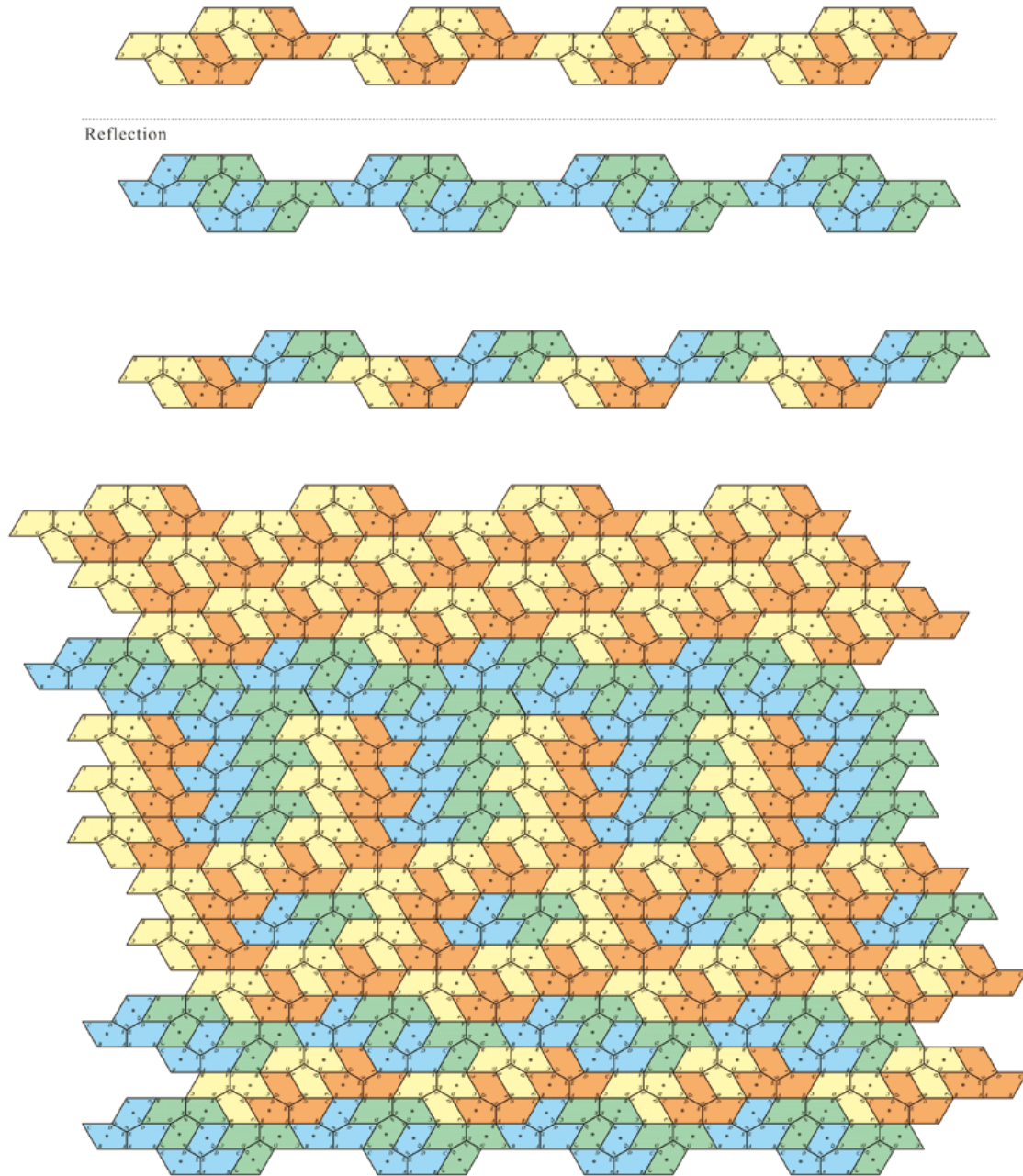


Figure. 32: Tiling with Class S1.

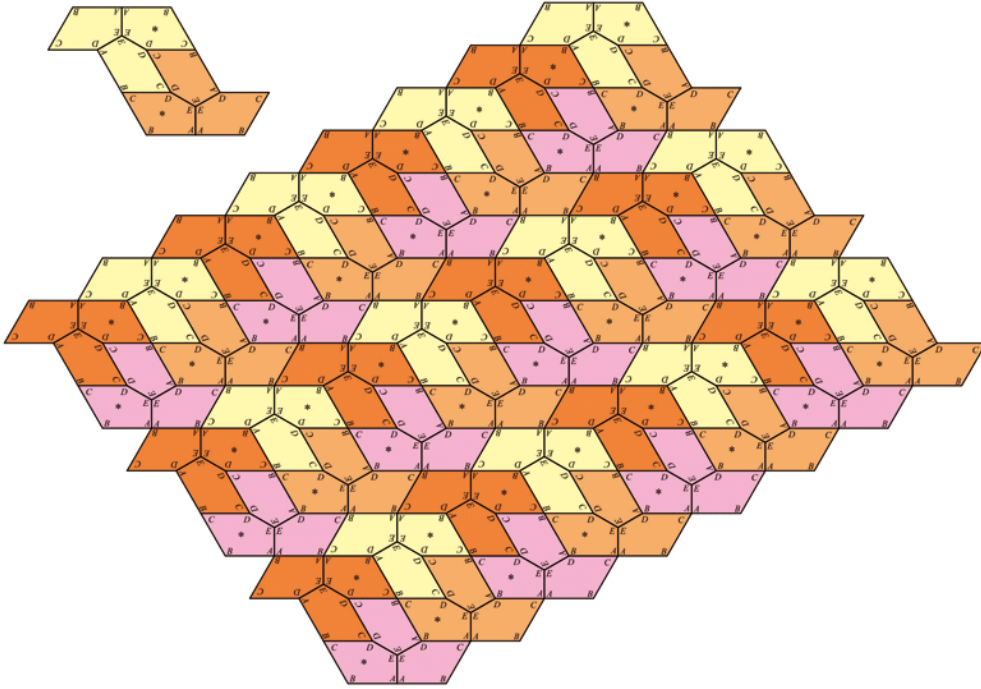


Figure. 33: Tiling with Class S2.

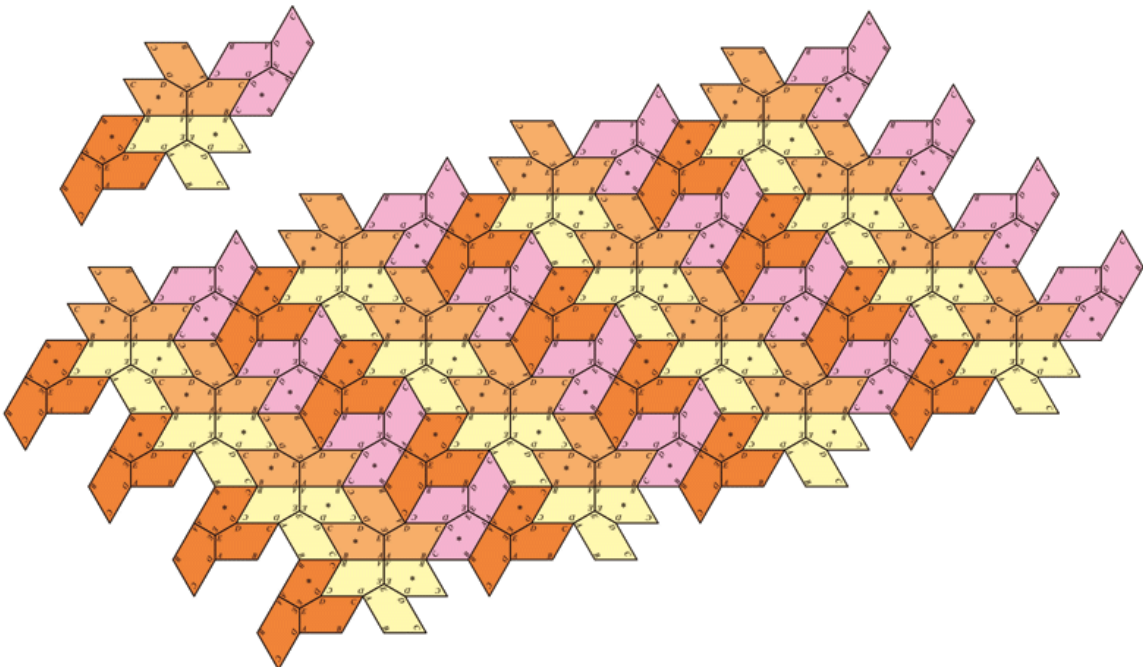


Figure. 34: Tiling with Class S3.

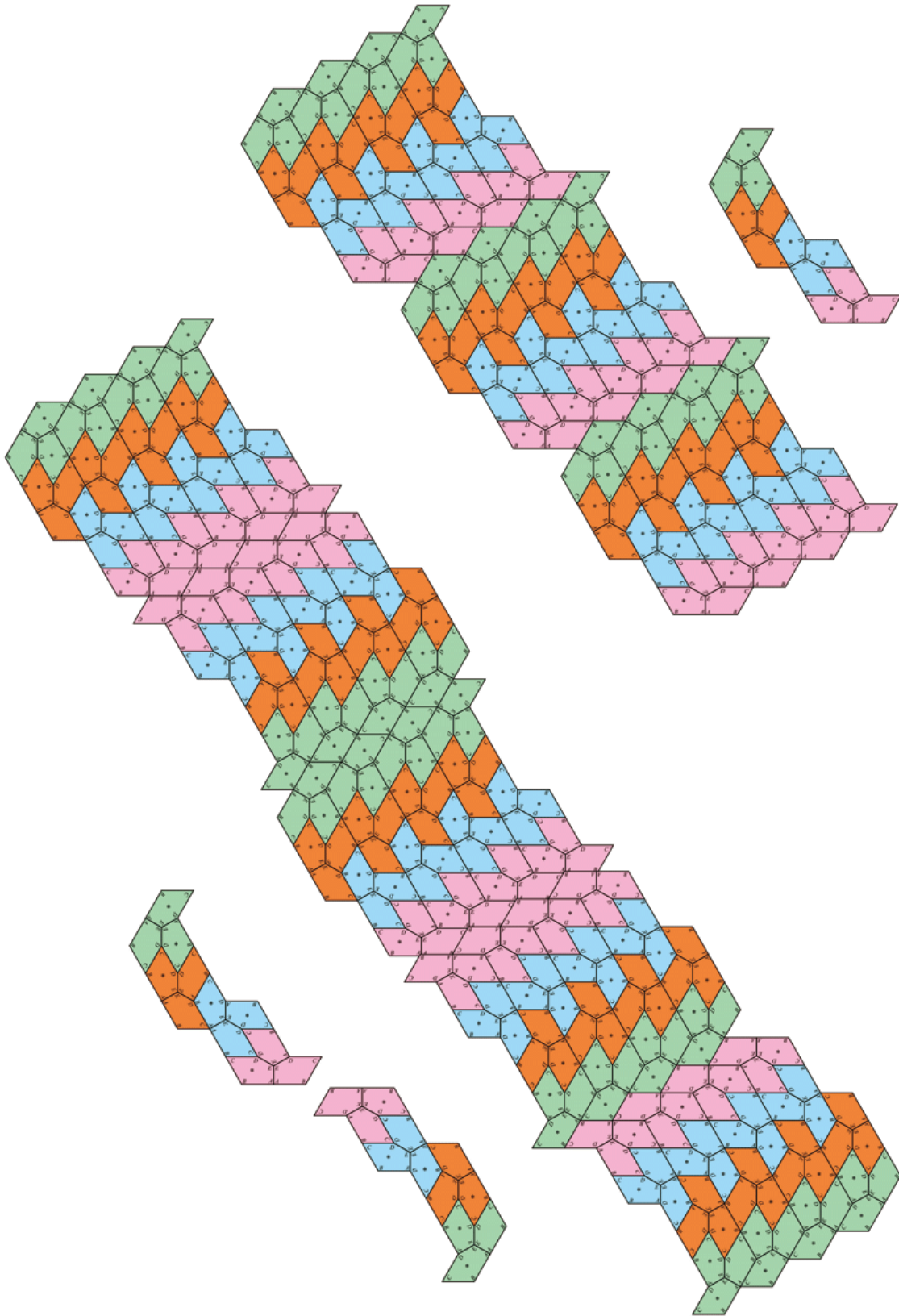


Figure. 35: Tiling with Class S4.

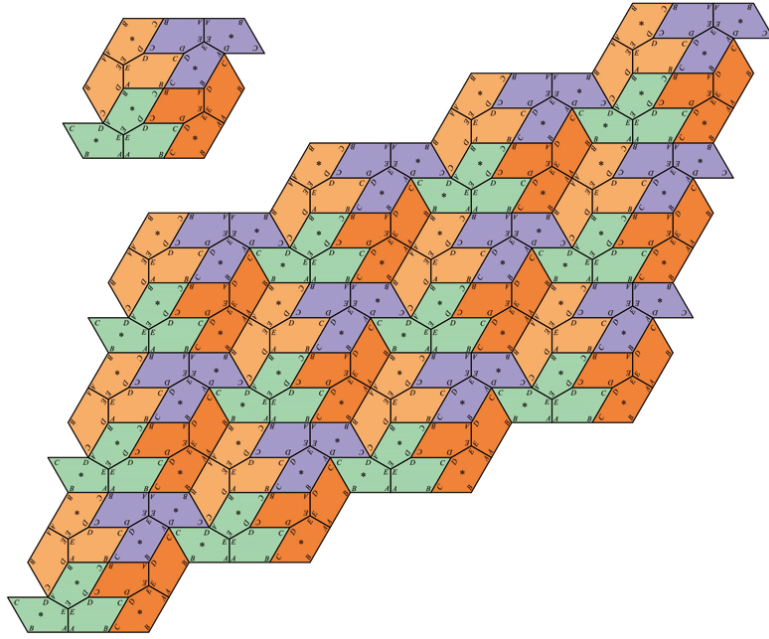


Figure. 36: Tiling with Class S4.

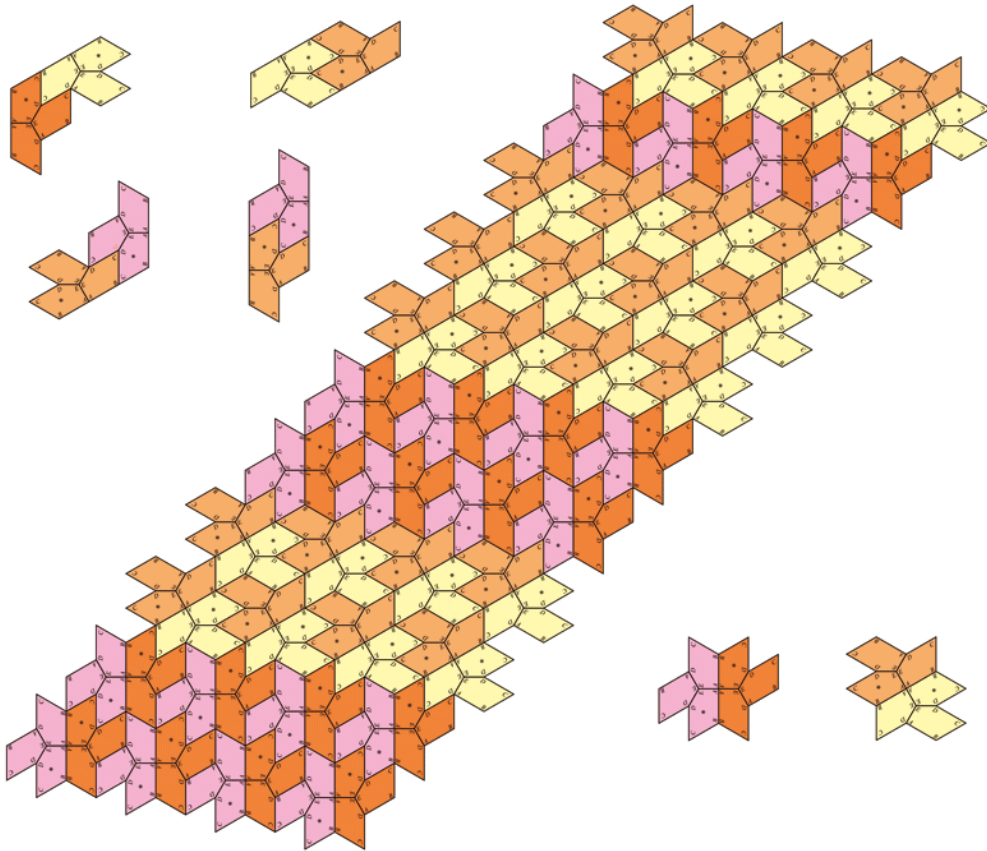


Figure. 37: Tiling with Classes S1 and S3.

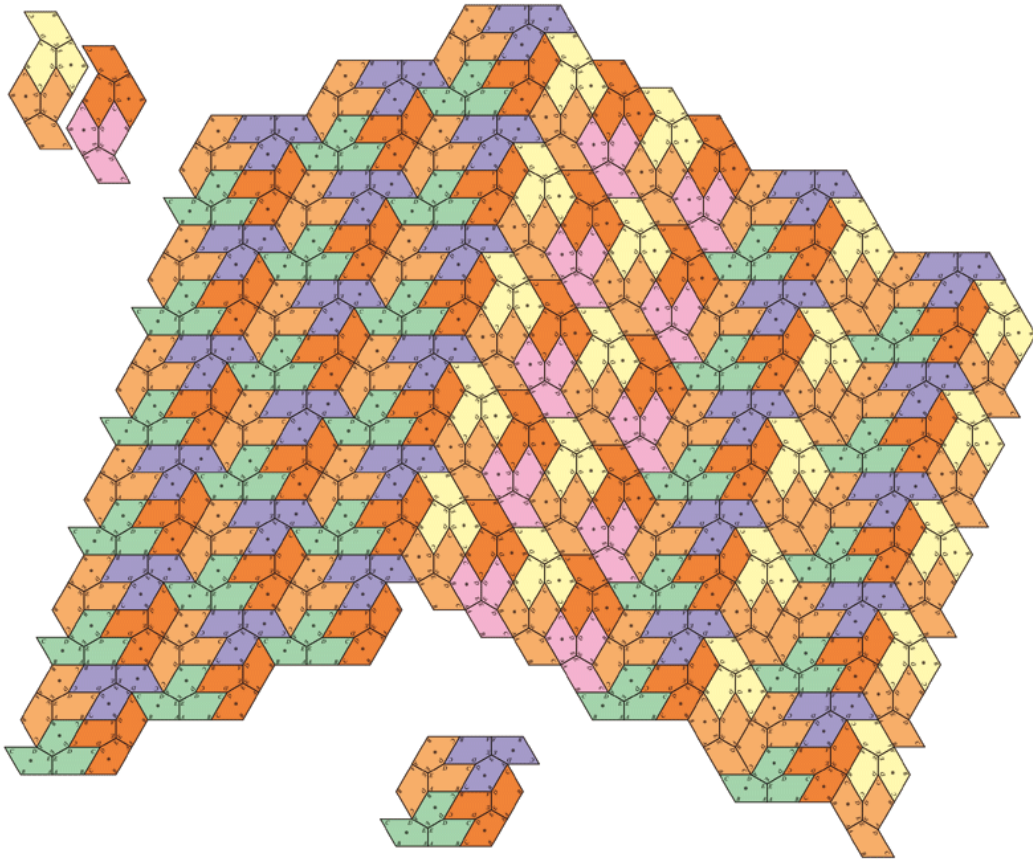


Figure. 38: Tiling with Classes S2 and S4.

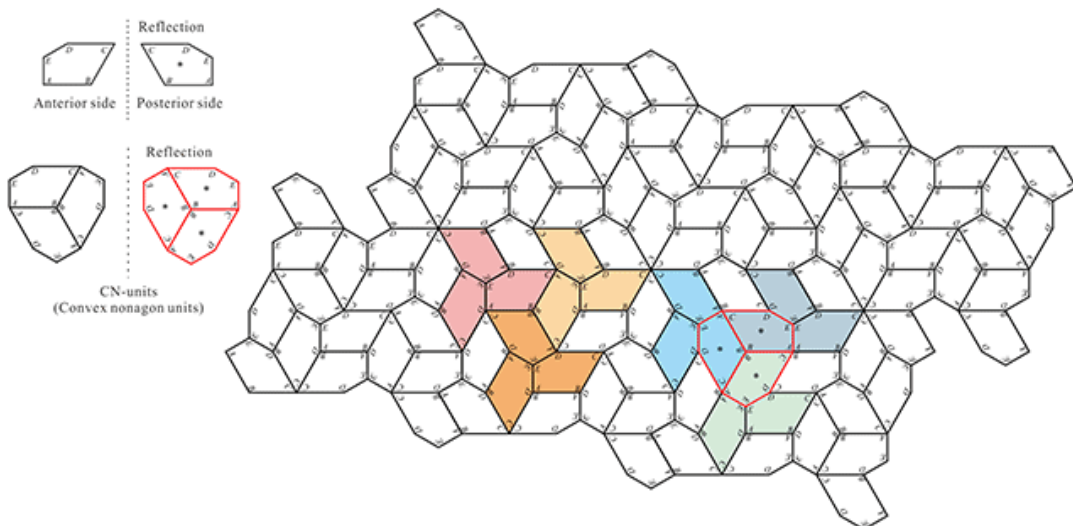


Figure. 39: Reversing of a CN-unit in Rice1995-tiling.

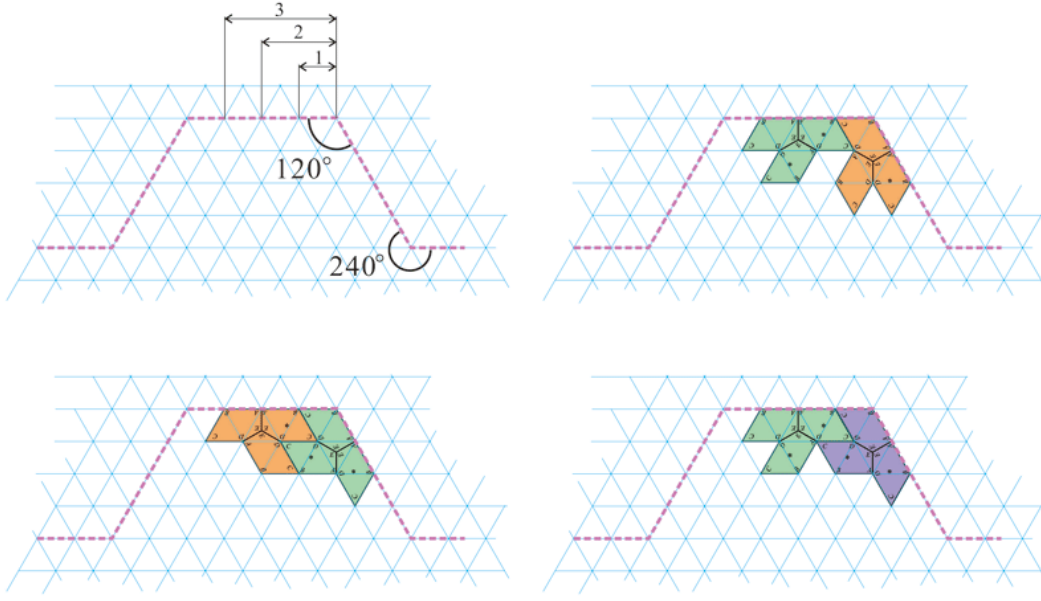


Figure. 40: Three patterns that use the two ship units.

5.3 Hexagonal flower L2 unit and tilings

Hereafter, the shape that is formed by 72 TH-pentagons in Figure 41 is referred to as a *hexagonal flowers L2 unit* (HFL2-unit). The length of the sides of HFL2-unit is twice the length of the sides of HFL1-unit. The arrangements of the internal TH-pentagons (or windmill units and ship units) that form HFL2-unit are 14 patterns in Figure 41, not distinguishing reflections and rotations (i.e., the number of unique patterns of HFL2-unit is 14). Note that it was confirmed by the program that there are 14 unique patterns.

Like HFL1-units, the HFL2-units can generate tilings by combining HFL2-units. The contiguity methods are three patterns as shown in Figures 42, 43, and 44. Note that the tilings in Figures 42, 43, and 44 are formed by an HFL2-unit of Figure 41(a) and its reflected image. (As for tilings in Figure 56 in the Appendix, they are cases of an HFL2-unit of Figure 41(m).) Of course, the parts of CN-units in tilings can be reversed freely, and different patterns of HFL2-units which contain reflections and rotations can be freely connectable in the one tiling (see Figure 45).

HFL2-units in Figure 41(a) and (n) contain a concentric HFL1-unit, and contain 18 windmill units and six ship units. HFL2-units in Figure 41(a)–(m) contain six CN-units. HFL2-units in Figure 41(m) are formed by 24 ship units. Therefore, a tiling by only the HFL2-units in Figure 41(m) (see Figure 56 in Appendix) is a tiling with CN-units by only the ship units³.

The outline of the fundamental region of the Rice1995-tiling appears as a unit in the tiling in Figure 43. Therefore, as shown in Figure 46 or Figures 57 and 58 in the Appendix, tilings with HFL1-units and tilings with HFL2-units are connectable.

³ If all the CN-units in the Rice1995-tiling in Figure 4 is reflected, it is a Rice1995-tiling by only the ship units. This tiling is equal to the tiling in Figure 56(b) in Appendix. That is, a tiling of HFL2-units according to the contiguity method in Figure 43 and Figure 56(b) is a Rice1995-tiling.

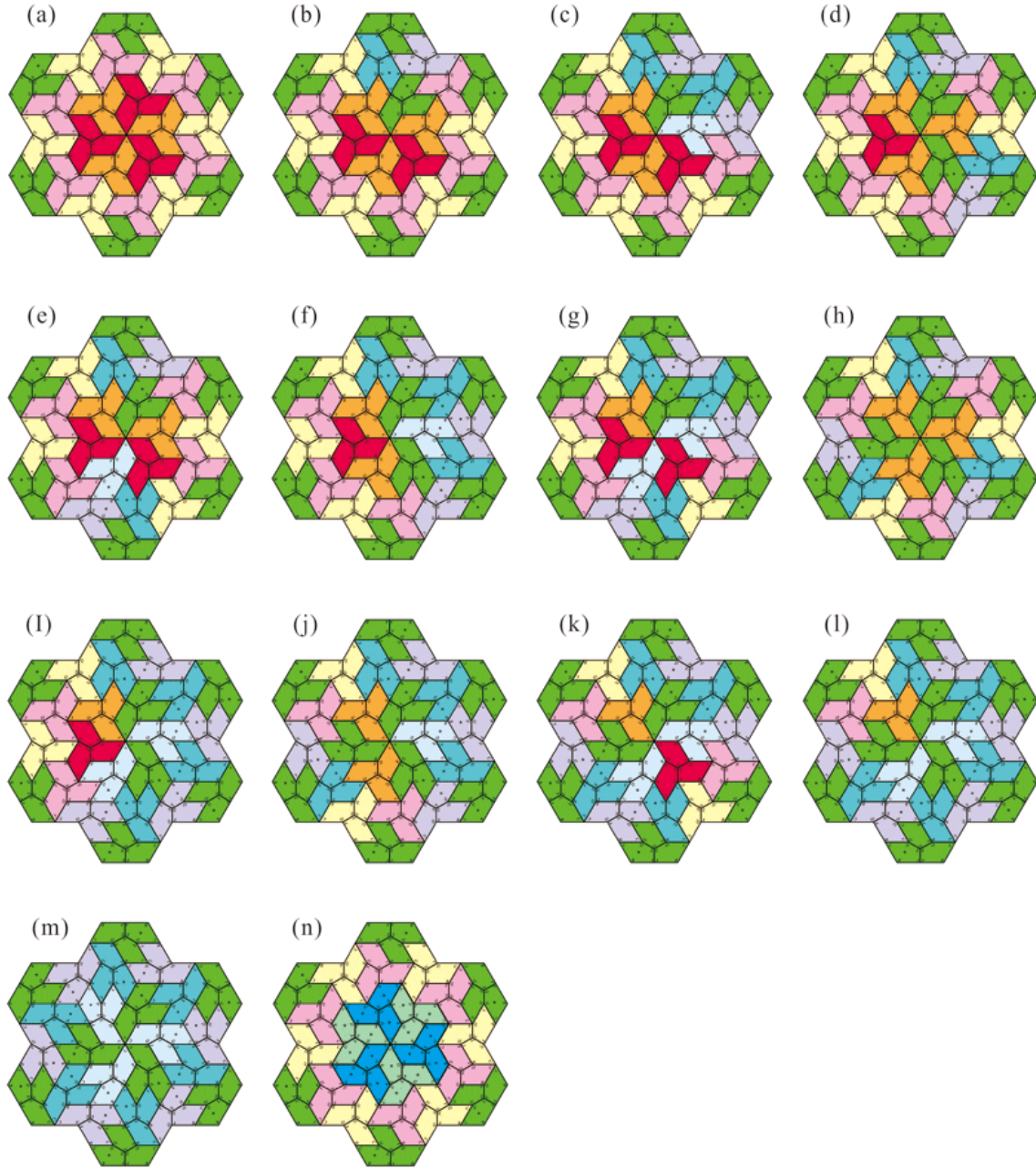


Figure. 41: HFL2-units of 14 unique patterns.

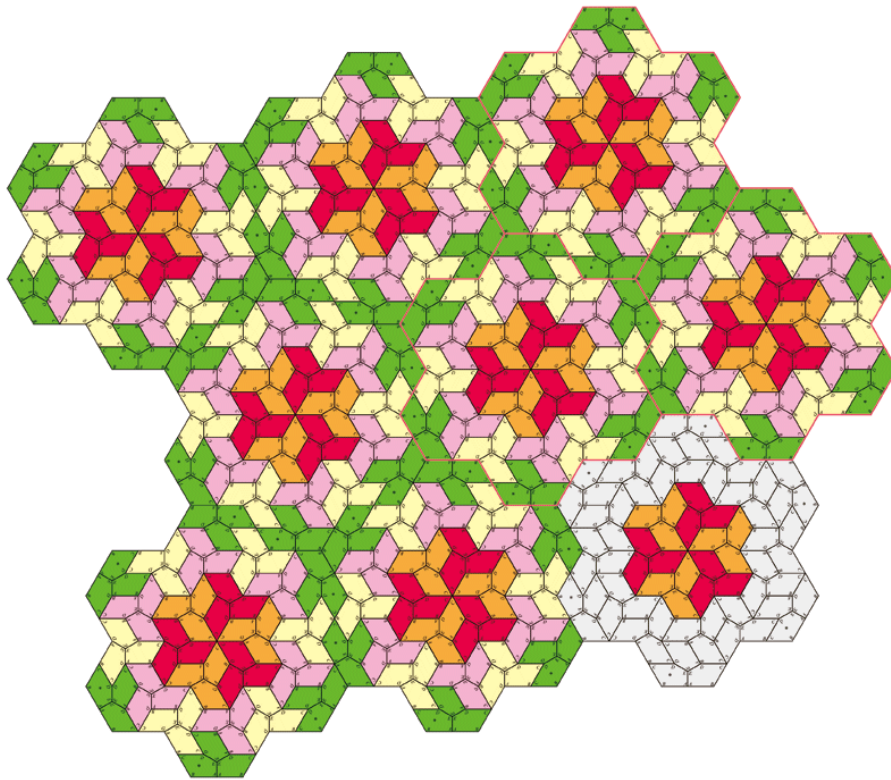


Figure. 42: Example of tiling by HFL2-units.

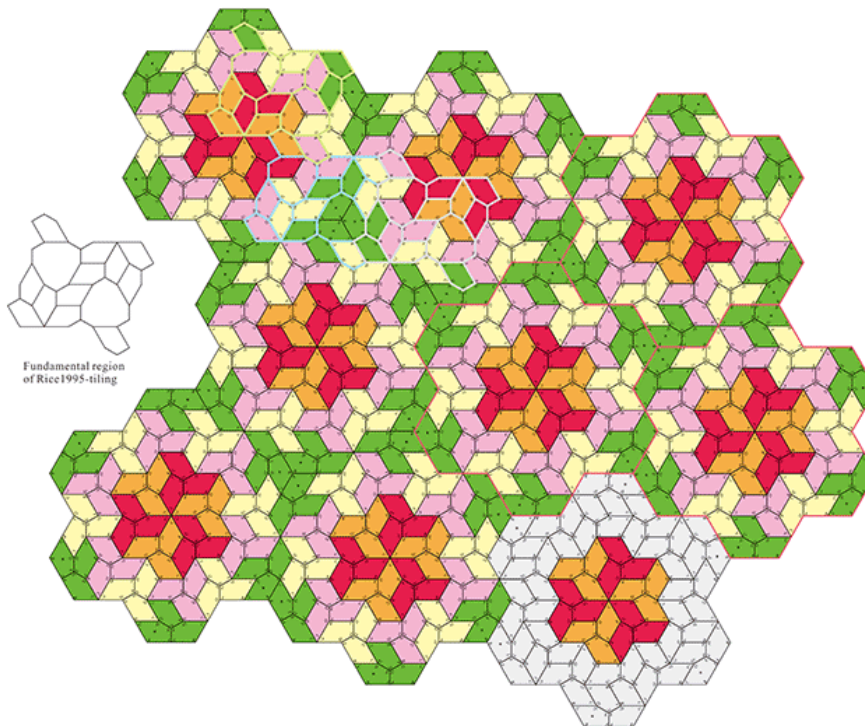


Figure. 43: Example of tiling by HFL2-units.

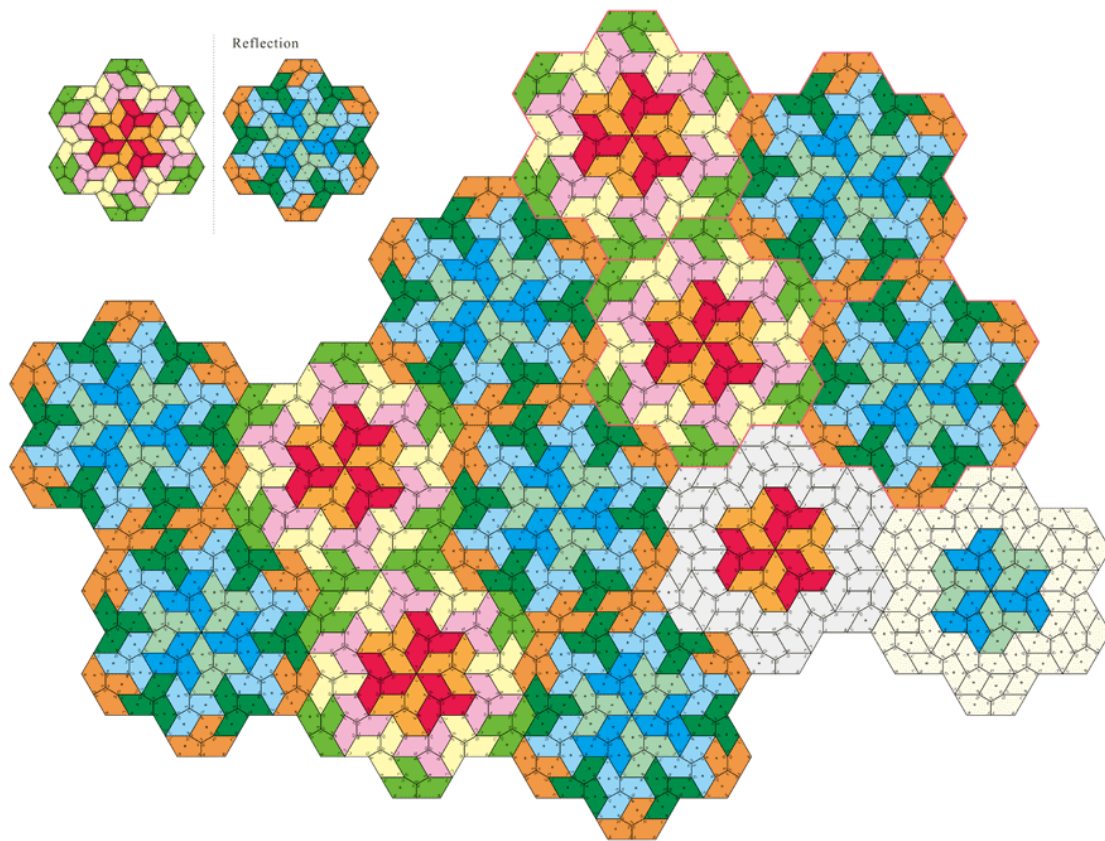


Figure. 44: Example of tiling by HFL2-units.

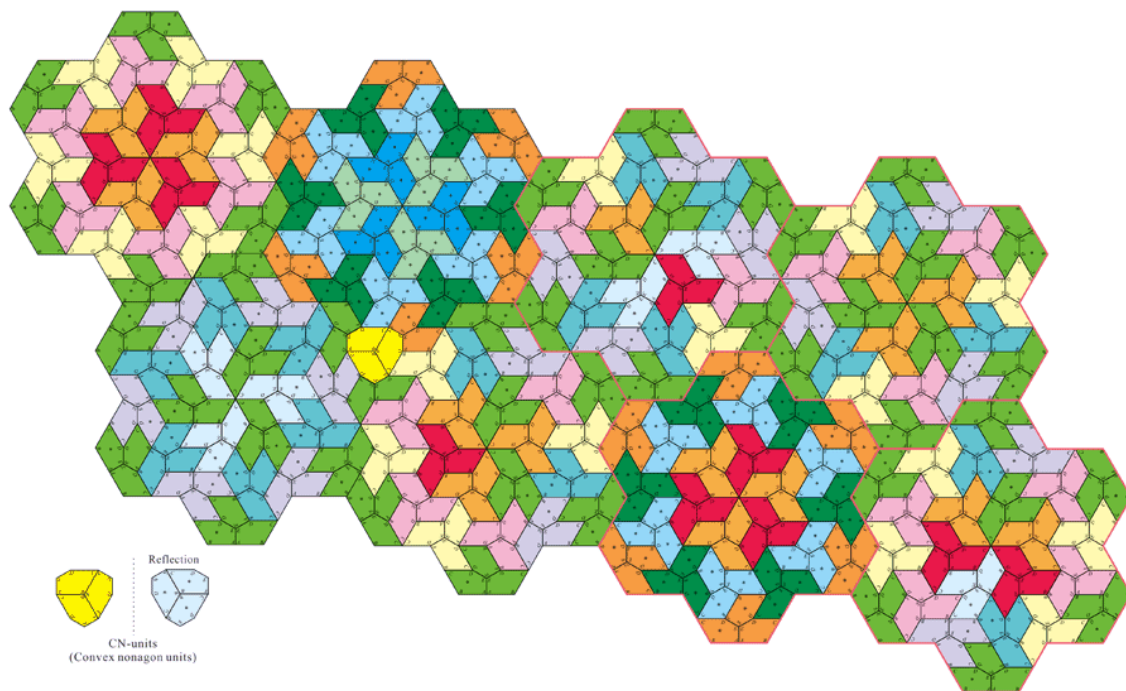


Figure. 45: Example of tiling by HFL2-units.

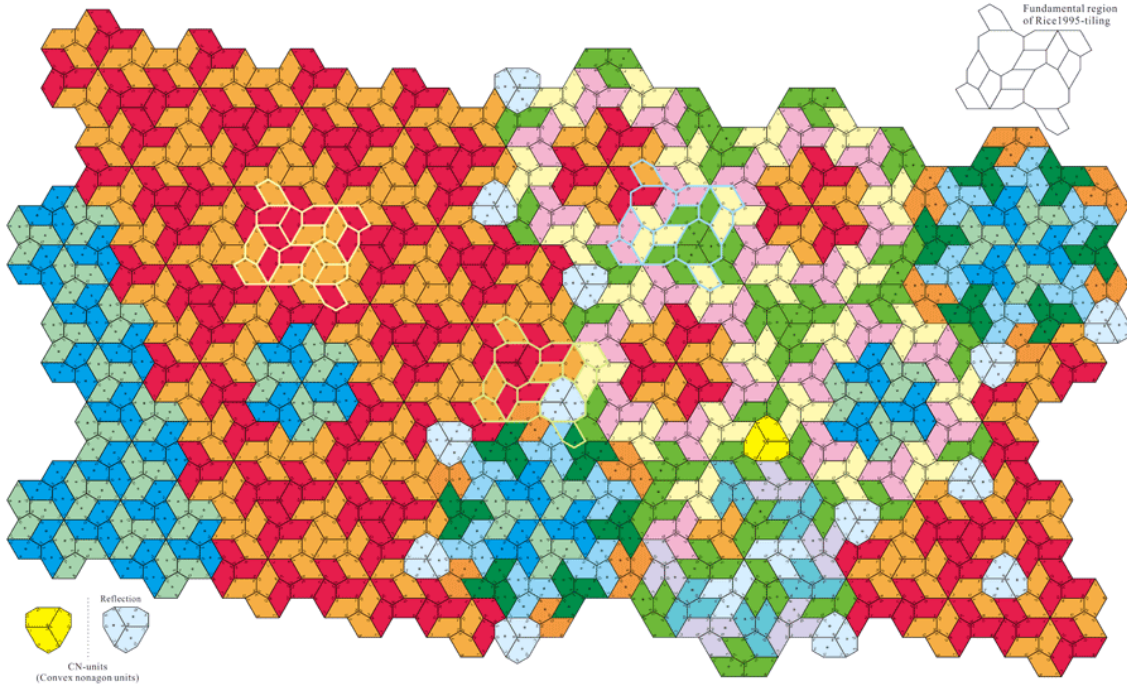


Figure. 46: Example of tiling by HFL1-units and HFL2-units.

5.4 Hexagonal flower L3 unit and tilings

Hereafter, the shape that is formed by 162 TH-pentagons in Figure 47 is referred to as a *hexagonal flowers L3 unit* (HFL3-unit)⁴. The length of the sides of an HFL3-unit is three times the length of sides of an HFL1-unit. The arrangements of the internal TH-pentagons (or windmill units and ship units) that form HFL3-unit are two patterns in Figure 47, not distinguishing reflections and rotations (i.e., the number of unique patterns of HFL3-unit is two). Note that it was confirmed by the program that there are two unique patterns. HFL3-units in Figure 47 contain a concentric HFL1-unit, and contain 18 windmill units and 36 ship units.

Like HFL1-units, the HFL3-units can generate tilings by combining HFL3-units. The contiguity methods are three patterns as shown in Figures 48, 49, and 50. Note that the tilings in Figures 48, 49, and 50 are formed by an HFL3-unit of Figure 47(a) and its reflected image. Then, the parts of CN-units in tilings can be reversed freely, and different patterns of HFL3-units which contain reflections can be freely connectable in the one tiling (see Figure 51).

5.5 Skewed hexagonal flower unit and tilings

From the inside of the HFL3-unit, take out the 18 sided polygon as shown in Figure 52. This 18-sided polygon is formed with side lengths of two and one. Therefore, this 18-sided polygon is called a *skewed hexagonal flower unit* (skewed HF-unit). The unique arrangement of the internal TH-pentagon (or windmill unit and ship unit) that forms a skewed HF-unit is of two types shown in Figure 52. The skewed HF-unit that contains 54 TH-pentagons, is made of an HFL1-unit, six windmill units, and twelve ship units.

⁴ The arrangement of the internal TH-pentagons of HFL3-units was discovered by Toshihiro Shirakawa.

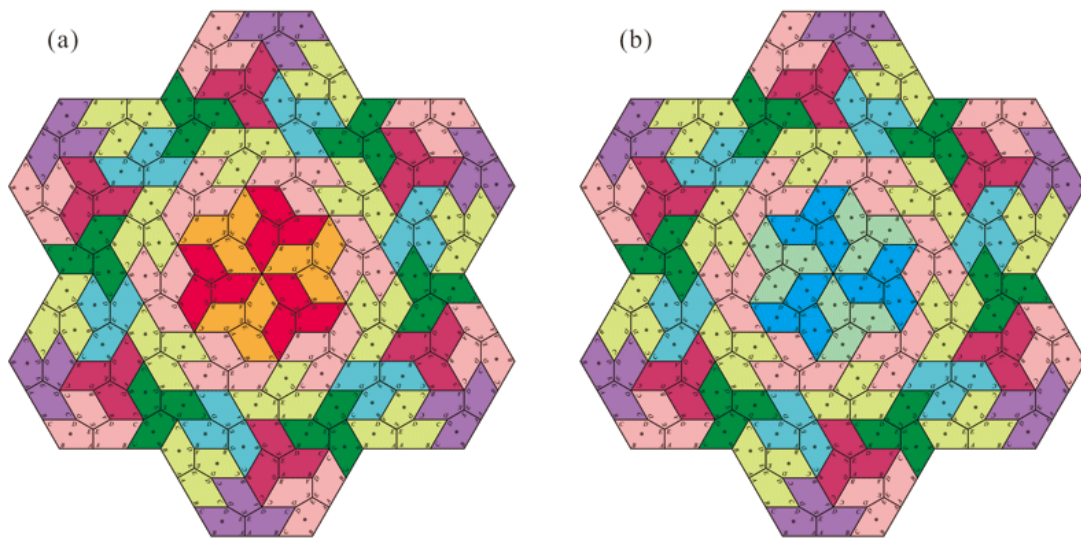


Figure. 47: HFL3-units of two unique patterns.

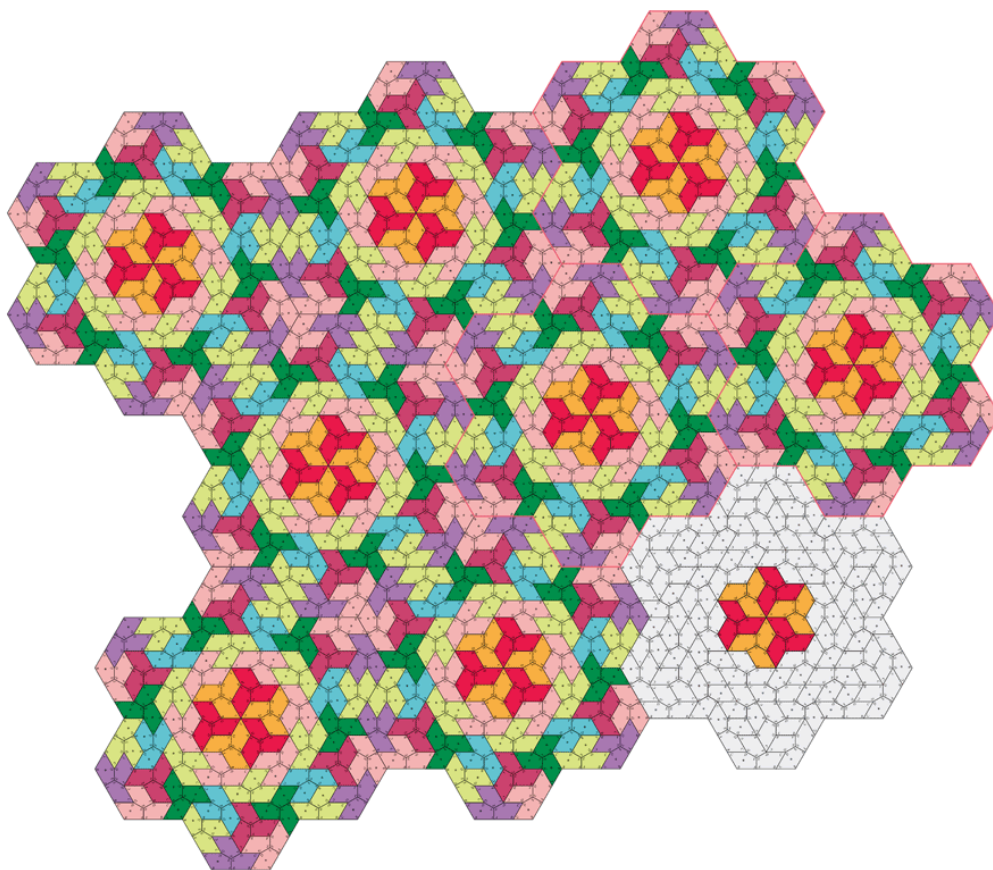


Figure. 48: Example of tiling by HFL3-units.

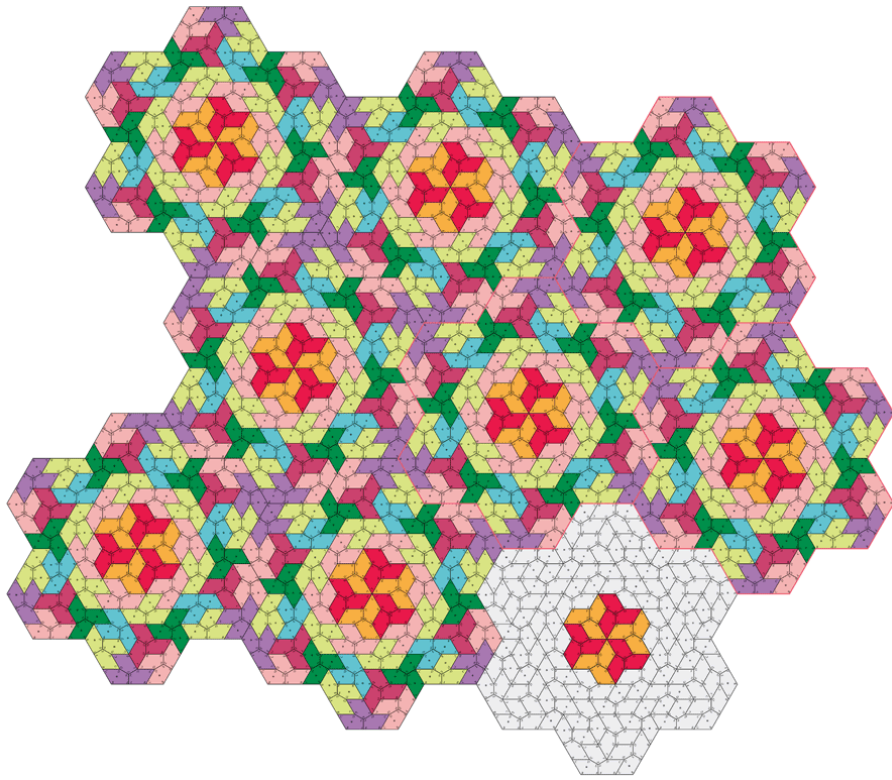


Figure. 49: Example of tiling by HFL3-units.

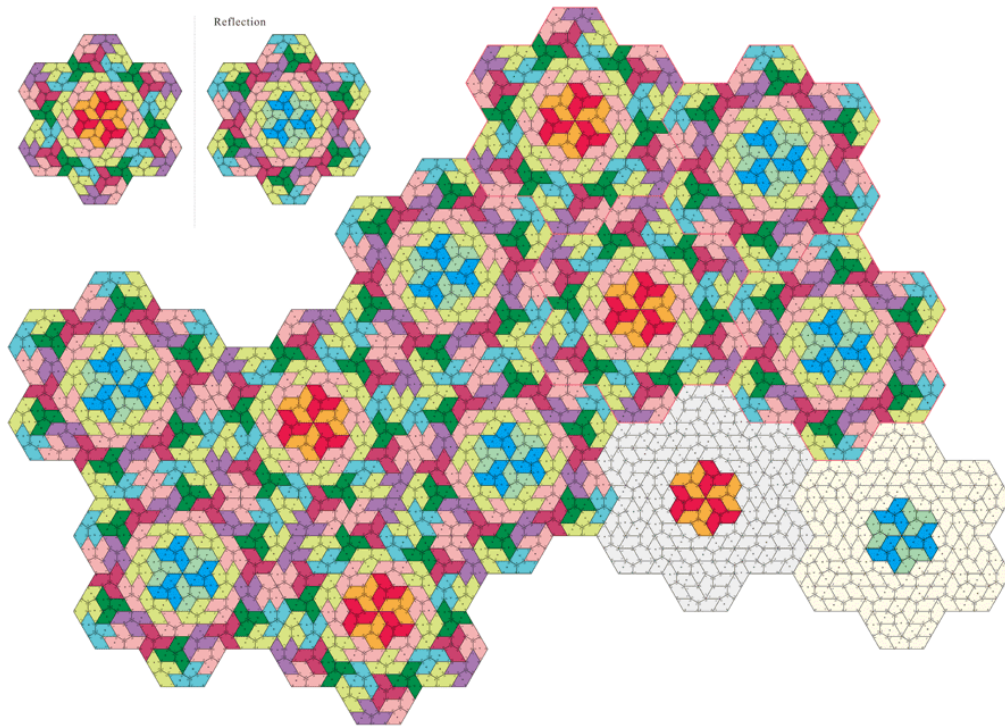


Figure. 50: Example of tiling by HFL3-units.

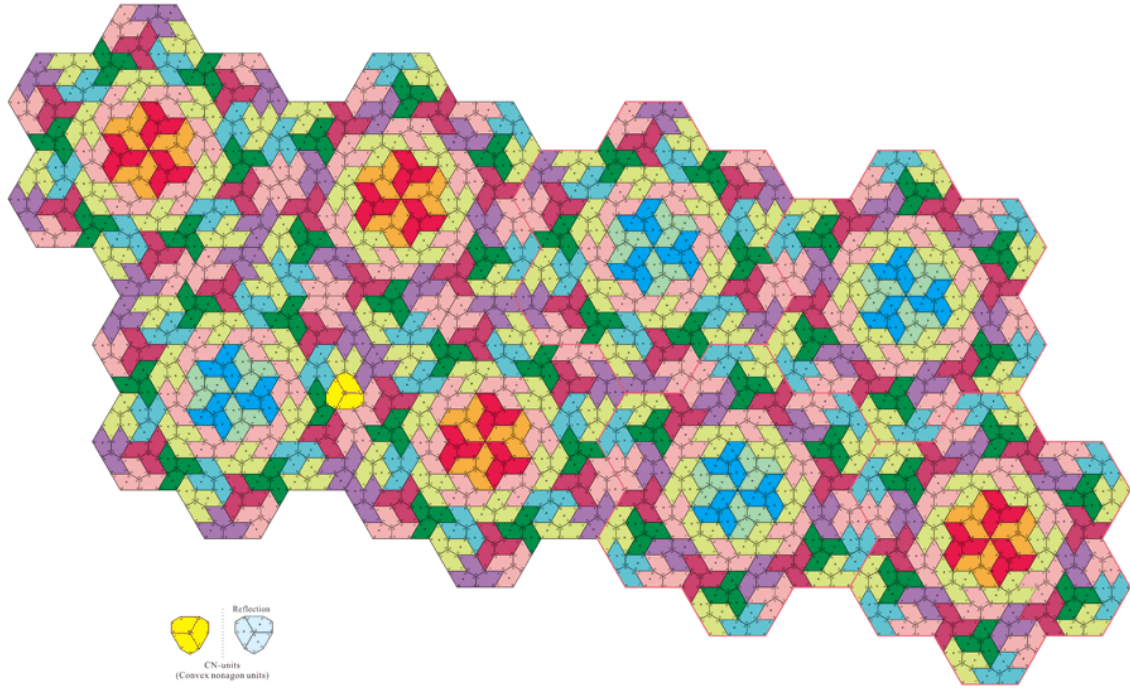


Figure. 51: Example of tiling by HFL3-units.

The skewed HF-unit has only one contiguity method for a tiling. However, the parts of CN-units in tilings can be reversed freely, and different patterns of skewed HF-units can be freely connectable in the one tiling (see Figure 53).

Using a program, it was confirmed that there is no TH-pentagon arrangement that fills a similar 18 sided polygon in Figure 52.

5.6 Tiling by the windmill units and the ship units based on the tiling with Classes S2 and S4

The tiling with Classes S2 and S4 in Figure 38 in Section 4 contains CN-units. Therefore, as shown in Figure 54, since the parts of CN-units in tilings can be reversed freely, the tiling of Figure 38 by only the ship units can be turned into a tiling with windmill units and ship units.

6 Conclusion

The authors identified novel properties of the tilings of TH-pentagon. As a result, many new tilings were found (see Sections 3, 4, and 5). The TH-pentagon admits many periodic tilings and nonperiodic tilings. Thus, the TH-pentagon can create infinite tilings.

Since the search has not ended, new tilings will be found in the future⁵. If a polyhex (n -hexes) based on hexagons can be filled by a windmill unit and a ship unit, there may be

⁵ After writing this manuscript, it was found that Johannes Hindriks presents tilings by heptiamonds at the site <http://www.jhhindriks.info/37/>. The authors confirm that there are tilings which they have not found yet. They are introduced in the following manuscript.

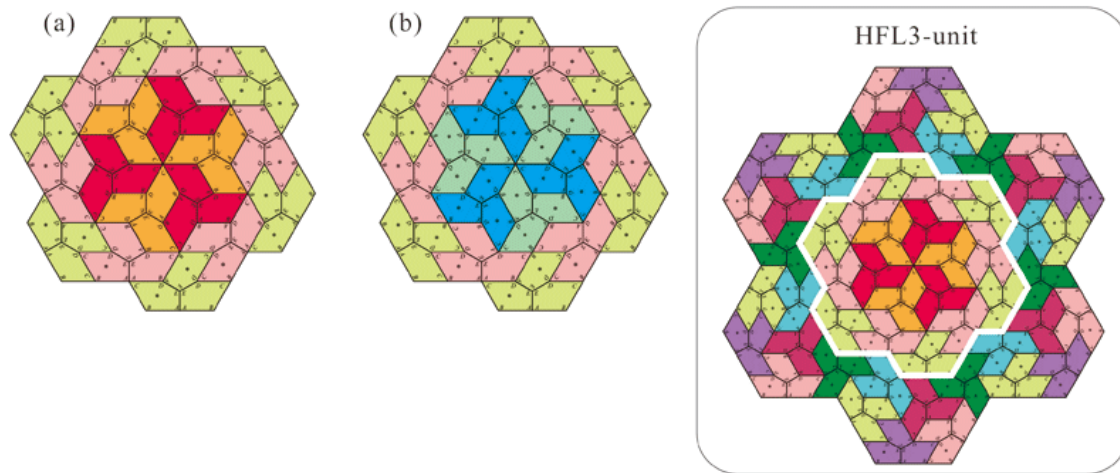


Figure. 52: Skewed HF-units of two unique patterns.

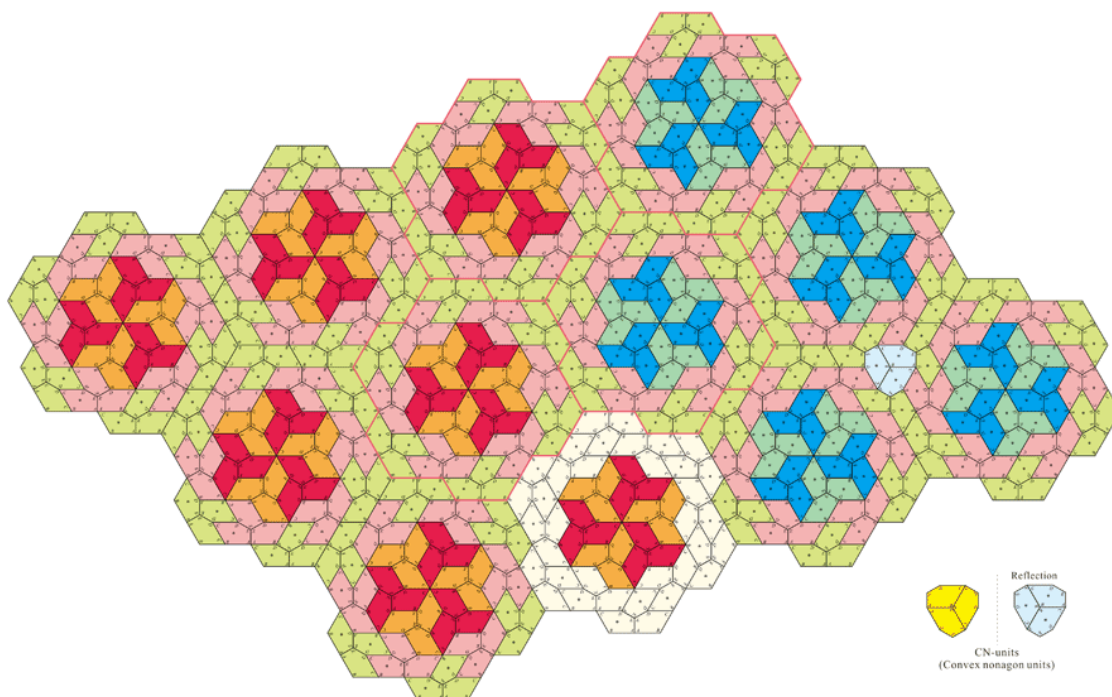


Figure. 53: Example of tiling by skewed HF-units.

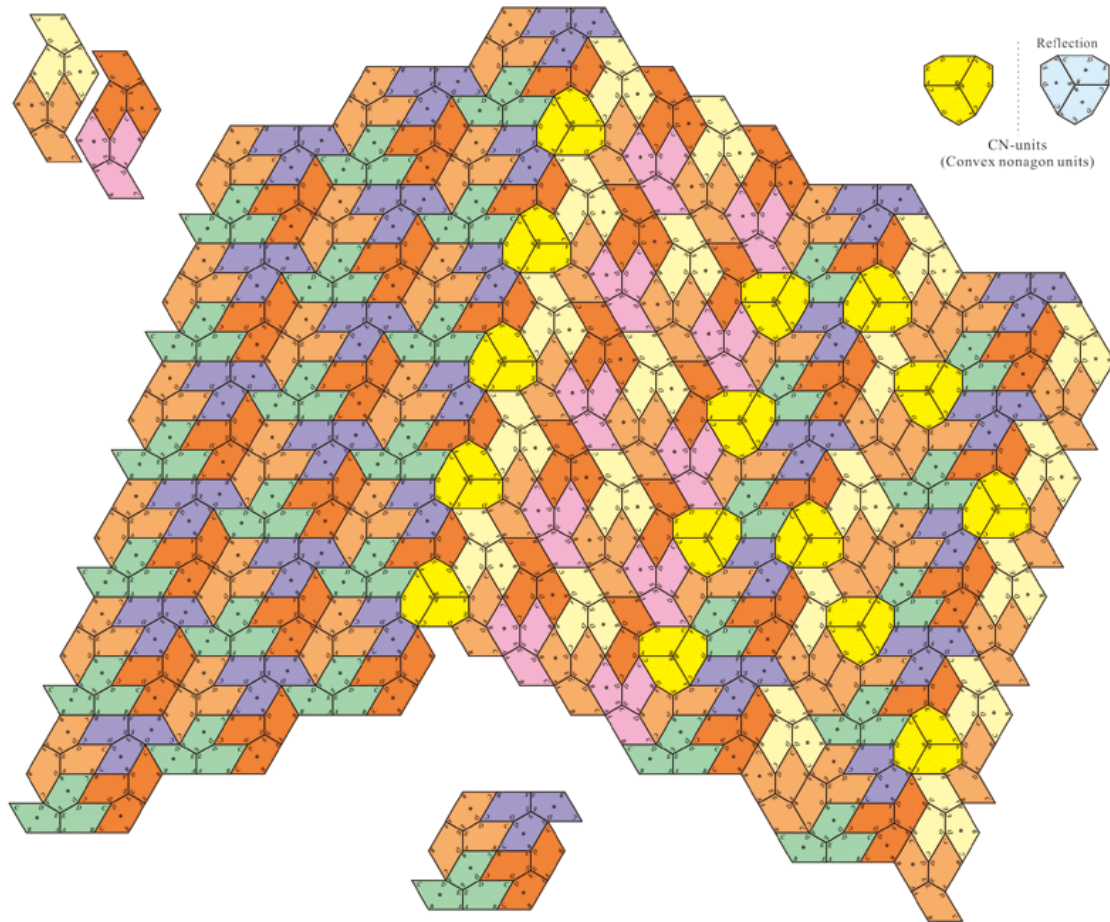


Figure. 54: Tiling by the windmill units and the ship units based on the tiling with Classes S2 and S4.

a new convex pentagon tiling [1, 15, 16]. The shape of a hexagonal flower unit corresponds to one of Heptahexa (7-hexes).

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Appendix

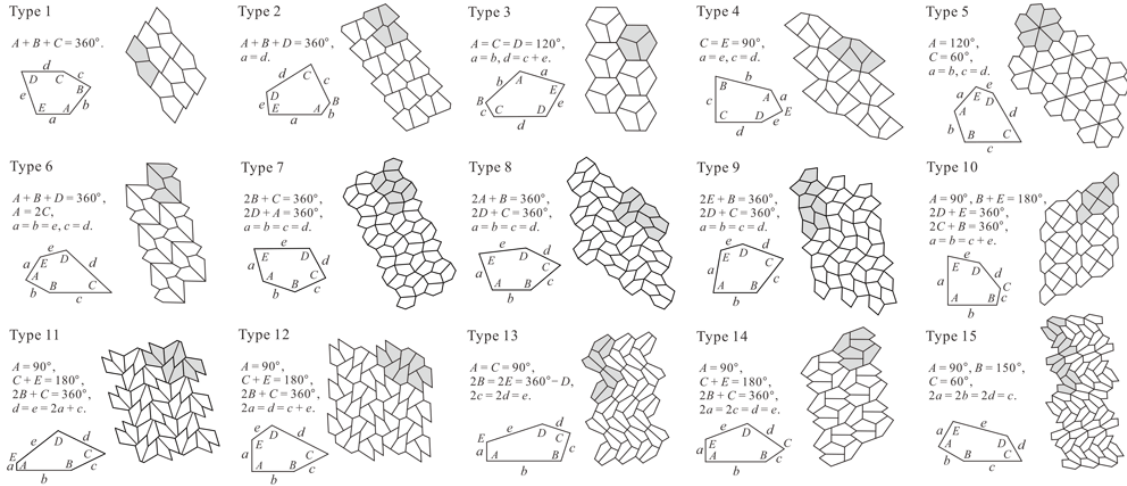


Figure. 55: Convex pentagonal tiles of 15 types. Each of the convex pentagonal tiles is defined by some conditions between the lengths of the edges and the magnitudes of the angles, but some degrees of freedom remain. For example, a convex pentagonal tile belonging to Type 1 satisfies that the sum of three consecutive angles is equal to 360° . This condition for Type 1 is expressed as $A + B + C = 360^\circ$ in this figure. The pentagonal tiles of Types 14 and 15 have one degree of freedom, that of size. For example, the value of C of the pentagonal tile of Type 14 is $\cos^{-1}((3\sqrt{57} - 17)/16) \approx 1.2099$ rad $\approx 69.32^\circ$.

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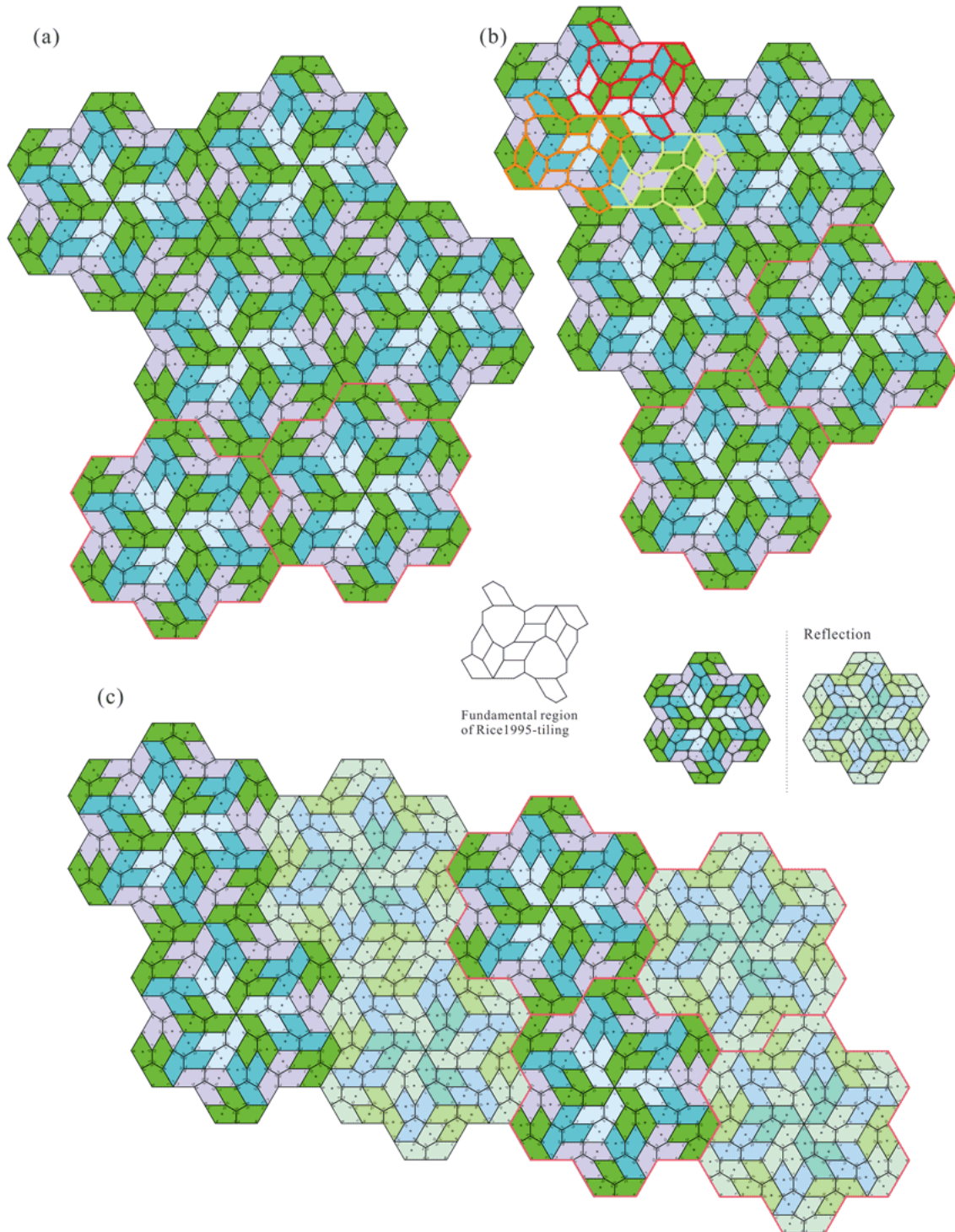


Figure. 56: Tilings by HFL2-unit of Figure 41(m) and its reflection image.



Figure. 57: Tiling by windmill units, HFL1-units, and HFL2-units.

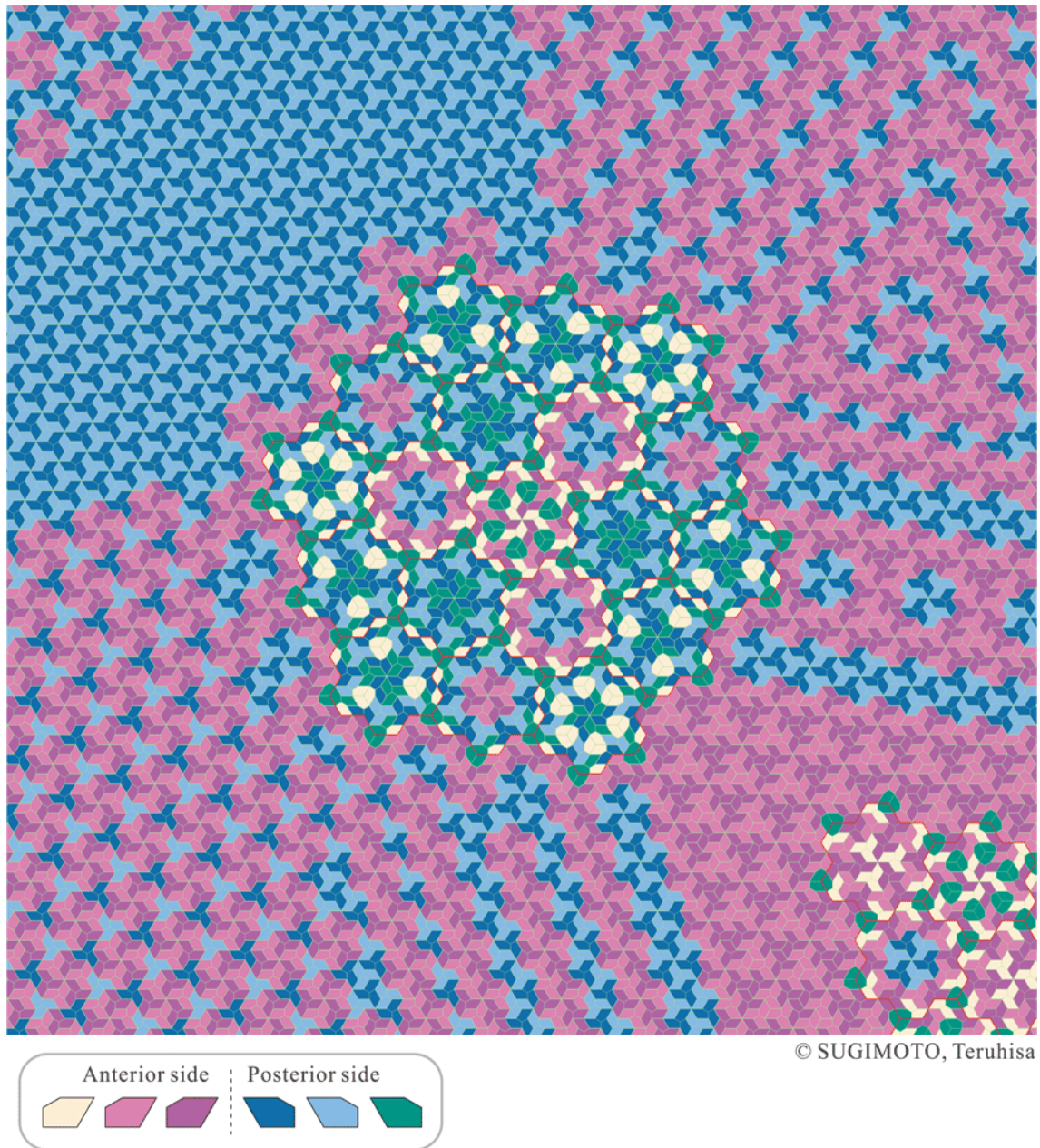


Figure. 58: Tiling by windmill units, HFL1-units, and HFL2-units

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